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# MIROSLAV FIEDLER, VLASTIMIL PTÁK

(MIROSLAV FIEDLER, VLASTIMIL PTÁK)

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**Abstract**

**Full Text**

**MIROSLAV FIEDLER, VLASTIMIL PTÁK**  
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## **ESTIMATES AND ITERATIVE METHODS FOR FINDING A SIMPLE EIGENVALUE OF AN ALMOST DECOMPOSABLE MATRIX**

*(Presented by Academician S. L. Sobolev on 6 III 1963)*

Let  $X$  be a complex linear space of dimension  $n$ . We shall regard the vectors of this space as row vectors  $x = (x_1, x_2, \dots, x_n)$ . A column vector with the same coordinates will be denoted by  $x'$ . Then the vectors  $y = (x_2, \dots, x_n)$  range over a space  $Y$  of dimension  $n - 1$ . In  $Y$  we prescribe a norm  $g$  (for example,  $g(y) = \sum_{i=2}^n |x_i|$ ), to which there corresponds the norm of a matrix (operator)  $B$  in  $Y$ ,  $g(B) = \sup g(yB)$  for  $g(y) = 1$ , and the "lower matrix norm"  $\hat{g}(B) = \inf g(yB)$  for  $g(y) = 1$ . As usual, for column vectors we introduce the conjugate norm  $g'(y') = \sup |yy'|$  for  $g(y) = 1$ .

For brevity we omit, in expressions of the type  $A - tE$ , where  $t$  is a number and  $E$  is the identity matrix, the symbol  $E$ . In studying the location of eigenvalues in the complex plane we use the following notation: if  $z_0$  is a complex number and  $\rho > 0$ , then by  $K(z_0, \rho)$  we denote the set of all complex numbers  $z$  for which  $|z - z_0| \leq \rho$ .

The following problem is considered in the paper. Let

$$A = \begin{pmatrix} a_{11} & a_1 \\ a'_2 & A_{22} \end{pmatrix}$$

be a block matrix (with blocks of dimensions 1 and  $n - 1$ ) such that at least one of the vectors  $a_1, a'_2$  is, in a certain sense, small. Then one may expect that some eigenvalue of the matrix  $A$  is close to  $a_{11}$ . Precise results are given in the following theorem:

**Theorem.** Let

$$A = \begin{pmatrix} a_{11} & a_1 \\ a_2 & A_{22} \end{pmatrix}$$

be a block complex matrix (with blocks of dimensions 1 and  $n - 1$ ), and let  $B = A_{22} - a_{11}$  be nonsingular. Suppose that for numbers  $a_1, a_2, a_3$  satisfying the inequalities

$$g(a_1)g'(a'_2)/\hat{g}(B) \leq a_2, \quad |a_1 B^{-1} a'_2| \leq a_1, \quad \hat{g}(B) \geq a_3,$$

the inequality

$$\sqrt{a_1} + \sqrt{a_2} < \sqrt{a_3}.$$

is fulfilled.

Then (in the notation indicated in item 3):

- 1) In the circle  $U^* = K(a_{11}; s_1 + \omega)$  there lies exactly one eigenvalue of the matrix  $A$ , which we shall denote by  $x$ .
- 2) The following three iterative processes are meaningful and convergent:

$$x_0 = a_{11}, \quad x_{k+1} = a_{11} - a_1(A_{22} - x_k)^{-1}a'_2, \quad \lim x_k = x; \quad (1)$$

$$y_0 = a_{11}, \quad y_{k+1} = a_{11} - \frac{a_1 B^{-1} a'_2}{1 + a_1(A_{22} - y_k)^{-1} B^{-1} a'_2}, \quad \lim y_k = x; \quad (2)$$

$$B_0 = B, \quad c'_0 = B_0^{-1} a'_2, \quad \omega_k = a_k c'_k, \quad B_{k+1} = B_k + c_k a'_1 + \omega_k; \quad (3)$$

$$c'_{k+1} = -\omega_k B_{k+1}^{-1} c'_k;$$

putting

$$z_k = a_{11} - \sum_{j=0}^{k-1} \omega_j$$

for  $k = 1, 2, \dots$ , we obtain that  $\lim z_k = x$ .

In the notation indicated in item 3), the inclusions

$$U^* \supset K(x_0; \rho_0) \supset K(x_1; \rho_1) \supset \dots,$$

$$U^* \supset K(y_0; r_0) \supset K(y_1; r_1) \supset \dots,$$

$$U^* \supset K(z_1; R_1) \supset K(z_2; R_2) \supset \dots$$

are valid, and the point  $x$  is the unique point of intersection of each of these three chains.

3) List of notation:

$$s_1 = \alpha_3 - \alpha_1 + \alpha_2, \quad s_2 = \alpha_3 + \alpha_1 - \alpha_2, \quad s_3 = \alpha_3 - \alpha_1 - \alpha_2,$$

$$w = \sqrt{s_1^2 - 4\alpha_2\alpha_3} = \sqrt{s_2^2 - 4\alpha_1\alpha_3} = \sqrt{s_3^2 - 4\alpha_1\alpha_2} > 0,$$

$$\lambda_i = \frac{s_i - w}{s_i + w} \quad \text{for } i = 1, 2, 3, \quad q = \frac{4\alpha_1\alpha_2}{s_3^2} < 1,$$

$$\rho_k = \frac{w\lambda_2\lambda_1^k}{1 - \lambda_2\lambda_1^k}, \quad r_k = \frac{w\lambda_2\lambda_3^k}{1 - \lambda_2\lambda_3^k}, \quad R_k = s_3 \frac{1}{2^k} q^{2k-1}.$$

**Corollary.** Suppose

$$A = \begin{pmatrix} a_{11} & a_1 \\ a'_2 & A_{22} \end{pmatrix}$$

is a block complex matrix (with blocks of sizes 1 and  $n-1$ ) and  $B = A_{22} - a_{11}$  is nonsingular. Let  $\gamma, \beta_1, \beta_2$  be real numbers such that

$$\hat{g}(B) \geq \gamma > 0, \quad g(A_1) \leq \beta_1, \quad g'(a'_2) \leq \beta_2.$$

Assume that

$$\gamma^2 > 4\beta_1\beta_2.$$

Then the circle

$$K \left( a_{11}; \frac{1}{2} (\gamma - \sqrt{\gamma^2 - 4\beta_1\beta_2}) \right)$$

contains exactly one eigenvalue of the matrix  $A$ .

This corollary is a strengthening of theorem (5,5) from paper (1).

The proof of the theorem is based on the following simple lemma:

**Lemma.** Suppose

$$A = \begin{pmatrix} a_{11} & a_1 \\ a'_2 & A_{22} \end{pmatrix}$$

is a square block complex matrix with blocks of sizes 1 and  $n$ , and suppose  $\lambda$  does not belong to the spectrum of the matrix  $A_{22}$ .

Then

$$\det(A - \lambda) = (a_{11} - \lambda - a_1(A_{22} - \lambda)^{-1}a'_2) \det(A_{22} - \lambda),$$

so that  $\lambda$  is an eigenvalue of the matrix  $A$  if and only if

$$a_{11} - \lambda = a_1(A_{22} - \lambda)^{-1}a'_2.$$

Using this lemma, for  $|t| < \hat{g}(B)$  we introduce the function

$$f(t) = -a_1(B - t)^{-1}a'_2.$$

It follows from the lemma that (under the condition that the matrix  $A_{22} - \lambda$  is nonsingular) the number  $\lambda$  is an eigenvalue of the matrix  $A$  if and only if  $\lambda - a_{11} = f(\lambda - a_{11})$ . Process (1) can evidently be rewritten in the form

$$x_0 - a_{11} = 0, \quad x_{k+1} - a_{11} = f(x_k - a_{11}).$$

Analyzing this process, we see that it converges to an eigenvalue of the matrix  $A$  lying in any  $K(x_k; \rho_k)$ . Since  $f(t_1) - f(t_2) = (t_1 - t_2)a_1(B - t_1)^{-1}(B - t_2)^{-1}a_2$ , in  $U^*$  there lies no more than one eigenvalue of the matrix  $A$ .

The proof of convergence and of the estimates for processes (2) and (3) is carried out analogously.

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## CITED LITERATURE

1. M. Fiedler, V. Ptak, *Czechoslovak Mathematical Journal*, **12** (87), 558 (1962).

*Note: Figure translations are in progress. See original paper for figures.*

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