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Abstract

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MATHEMATICS

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ON LINEAR STRUCTURES WITH A SUFFICIENT SET OF MAXIMAL l-IDEALS

(Presented by Academician A. I. Mal' tsev, 5 I 1963)

Some properties are studied of Archimedean* K-lineals (linear structures) in which every l-ideal is embedded in a maximal one.

In the theory of l-groups and, in particular, K-lineals, the concept of an l-ideal plays a significant role. We shall everywhere have in mind only **proper** l-ideals. There is a certain analogy between ideals in rings (and algebras) and l-ideals in l-groups and K-lineals. It is well known that the presence of a multiplicative identity in a ring (or algebra) ensures the possibility of embedding every ideal in a maximal one. In the case of a K-lineal (or l-group), the analogous property for l-ideals is ensured by the presence of a strong unit**. In the general case this is, generally speaking, not so. For example, in the Archimedean K-lineal of all functions continuous on $(0, 1]$, the subset of bounded functions is an l-ideal that is not embedded in any maximal one.

Let us introduce the basic definition. We first recall that an l-ideal, or normal sublineal, of a K-lineal X is any linear subset $I \subset X$ such that, if $x \in I$, $y \in X$, and $|y| \leq |x|$, then $y \in I$ (I is automatically a substructure of X). We shall use only the first of these terms.

Definition. An Archimedean K-lineal will be called a K-lineal with a sufficient set of maximal l-ideals, or an MK-lineal, if every one of its l-ideals can be embedded in a maximal one. It is clear that a similar definition can also be given for an arbitrary l-group.

It is MK-lineals that are mainly studied in this work. All K-lineals occurring below will, without any reservations, be considered Archimedean. In particular, by a K-lineal of bounded elements we shall mean an Archimedean K-lineal with a strong unit.

As was noted above, a K-lineal of bounded elements is an MK-lineal. Thus the concept of an MK-lineal is a generalization of the concept of a K-lineal of bounded elements. In particular, an MK-lineal may even have no unit.

Let us note that in the case when a K-lineal is simultaneously an algebra, the theory of ordinary l-ideals differs essentially from the theory of l-ideals that are at the same time ideals of the algebra (or, more simply in the present case, ring

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ideals). Such ideals were studied, for example, in ([5]), and there they were also called l-ideals. In particular, it turns out

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* A K-lineal X is called Archimedean if from $x, y \in X$ and $0 \leq nx \leq y$ for every natural n it follows that $x = 0$.

** An element 1 of an l-group X is called a unit if $1 \wedge |x| > 0$ for every $x \neq 0$, $x \in X$; a unit is called strong if $|x| \leq n1$ for some natural $n = n(x)$. A strong unit belongs to no proper l-ideal, and this makes it possible to apply Zorn' s theorem to the set of all proper l-ideals ordered by inclusion; from the latter it follows that every l-ideal is embeddable in a maximal one.

different classes of K-lineals—algebras having a sufficient set of maximal l-ideals in the usual sense and in the sense of [(5)].

We shall use mainly the terminology of [(4)]. Some concepts introduced in [(2)] will also be used.

1. Let X be a K-lineal, $\hat{X}$ its K-completion, i.e., a K-space which is the completion of X by the method of sections ([(4)], p. 126). It is well known that $\hat{X}$ is realized as a certain normal* set of continuous functions on a uniquely determined, up to homeomorphism, extremally disconnected bicompact space Q, which may take the values $\pm\infty$ on nowhere dense subsets of Q ([(4)], p. 155). Therefore X too is realized as a certain set $\mathfrak{C}_\infty(Q)$ of such functions (not normal, if X is not a K-space). This bicompact space $Q = Q(X)$ in [(2)] was called the **canonical bicompact space** for X. The main result of this part is a characterization of an MK-lineal in terms of its realization in the form $\mathfrak{C}_\infty(Q)$ on $Q(X)$.

Preliminarily, using the terminology and results of [(2)], we introduce some concepts.

On $Q(X)$ there may be points t' such that for every function from $\mathfrak{C}_\infty(Q)$

$$t' \in \overline{\{t \in Q : x(t) \neq 0\}}.$$

For any such point, under every realization of X on $Q(X)$,** one always has $x(t') = 0$ for each of the functions considered in this realization—images of elements of X. It is clear that such points are, in a certain sense, superfluous on $Q(X)$.

Definition. The set $S = S(X)$ of all points of the canonical bicompact space which do not possess the property just considered, with the topology induced from $Q(X)$, will be called the **proper space** for X. In the case where X is a K-space, this concept coincides with the concept of proper space from [(8)].

The proper space $S(X)$, generally speaking, is smaller than the canonical bicompact space $Q(X)$, coinciding with it, as is not difficult to verify, if and only if

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X has a unit.

Let $t \in S(X)$. Then there exists $x_0 > 0$ such that, for some realization, $x_0(t) > 0$. The ratio $\dfrac{x(t)}{x_0(t)}$ has the same value for all realizations for which $x_0(t) \neq 0, \infty$; in [(2)] the numerical value of this ratio was denoted by $\dfrac{x}{x_0}[t]$.

Definition. A point $t_0 \in S(X)$ will be called **strong** if, for some $x_0 > 0$ and every $x \in X$, one always has

$$\left|\frac{x}{x_0}\right|[t_0] < +\infty.$$

The element x_0 itself is called a **strong local unit** (s.l.u.) at t_0. If there is no such x_0, the point will be called **weak**.

We shall call a K-lineal X a K-lineal with a **sufficient set of strong local units** (s.s.s.l.u. K-lineal) if $S(X)$ consists only of strong points (and, consequently, for every $t \in S(X)$ there is an s.l.u.).

Theorem 1. *For every strong point t_0, the set*

$$M_0 = \left\{x \in X : \frac{x}{x_0}[t_0] = 0\right\},$$

where x_0 is an s.l.u. at t_0, is a maximal l-ideal of the K-lineal X. Conversely, every maximal l-ideal in X is representable in this form.

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* A set of continuous functions is called **normal** if, together with every continuous function, it contains also every other function not larger in modulus.

** Different realizations of X on $Q(X)$ are possible. Each realization is determined by the choice of an element from a maximal extension (universal completion according to [(8)]) $\hat{X}$, which is realized as the function identically equal to one.

Theorem 2. *In order that a K-lineal be an MK-lineal, it is necessary and sufficient that it be a d.m.s.l.e. K-lineal.*

2. Here we formulate two theorems concerning the relation between maximal l-ideals in a K-lineal X and in its K-completion $\hat{X}$.

Theorem 3. *In order that a K-lineal X be an MK-lineal, it is necessary and sufficient that its K-completion $\hat{X}$ be an MK-lineal.*

It is not hard to see that always $S(\hat{X}) = S(X)$. Let now M_0 be a maximal l-ideal of X, considered in Theorem 1, and let $\hat{M}_0$ be the corresponding maximal l-ideal

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in $\hat{X}$. The natural correspondence $\hat{M}_0 \to M_0$ associates with each maximal l-ideal of $\hat{X}$ a maximal l-ideal in X. In this case $M_0 = \hat{M}_0 \cap X$. However, this correspondence need not be one-to-one (for $t_1 \neq t_0$ always $\hat{M}_1 \neq \hat{M}_0$, while at the same time it may happen that $M_1 = M_0$, i.e. the correspondence established in Theorem 1 between maximal l-ideals and strong points is, generally speaking, not single-valued if X is not a K-space). If the correspondence $\hat{M}_0 \to M_0$ is nevertheless one-to-one, we shall say that the sets of maximal l-ideals in X and in $\hat{X}$ coincide.

Let us recall a few more definitions. A **component** of a K-lineal X is any set $X' \subset X$ which is the disjoint complement of some $E \subset X$. It is easily proved ([3], p. 13; see also [6], p. 352) that this definition is equivalent to the usual definition of a component in a K-space ([4], p. 97). If for every $x \in X$ and for every component X' there exists a projection of x onto X' (for $x > 0$ this means the existence of $\sup\{x' \in X' : 0 \leq x' \leq x\}$), then X is called a K-lineal with projections.

Theorem 4. *Let X be an MK-lineal. In order that the sets of maximal l-ideals in X and in $\hat{X}$ coincide, it is necessary and sufficient that X be a K-lineal with projections.**

3. It is known (see ([1]), Theorems 2 and 3; ([3]), Theorems 3.5 and 3.6) that for a K-lineal of bounded elements X the following assertions are equivalent.

A. For some K-space Y, the spaces $(X \to Y)_r$ and $(\hat{X} \to Y)_r$ of regular operators from X to Y and from $\hat{X}$ to Y coincide (i.e. every regular operator from X to Y admits a unique extension to all of $\hat{X}$).

B. For every Y, $(X \to Y)_r$ and $(\hat{X} \to Y)_r$ coincide.

C. For some Y, the following criterion of disjointness of regular operators from $(X \to Y)_r$ is satisfied.

If $U_1, U_2 \in (X \to Y)_r$, then, for their disjointness, it is sufficient that there exist two mutually complementary components X_1 and X_2 such that

$$U_1(X_1) = U_2(X_2) = \{0\}.$$

D. For every Y in $(X \to Y)_r$, the criterion of disjointness is satisfied.

E. X is a K-lineal with projections.

In the case of an arbitrary K-lineal, assertion E implies all the others, but follows from none of them.

However, for an MK-lineal X the corresponding conclusion is valid; that is, we have

Theorem 5. *For an MK-lineal X, assertions A, B, C, D, E are equivalent.*

Along the way the following lemma is established, which is also of independent interest.**

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* Some characteristics of arbitrary K-lineals with projections are given in ([2,6]).

** For a K-lineal of bounded elements, the assertion of the lemma was essentially established in ([7]).

Lemma. Let X be a K-lineal, let x_0 be a weak order unit, and let $t_0 \in S(X)$. Then the functional

$$f_{t_0,x_0}(x) = \frac{x}{x_0}[t_0]$$

is discrete*. Conversely, every positive discrete functional has the indicated form.

4. As the simplest example of an MK-lineal Z that is not a K-lineal of bounded elements, let us take the so-called **Kaplansky K-lineal**—the set of all functions on $[0,1]$, each of which is continuous except possibly for a finite number of jumps of the second kind. It can be shown that every maximal l-ideal in Z is the set of all functions from Z that are finite at a certain point. It can also be shown that all functions from an arbitrary l-ideal are finite (or even equal to 0) at at least one point. Hence it is easily seen that Z is an MK-lineal.

Z does not possess some of the properties that hold in the K-lineal of bounded elements. In Z, as is not difficult to verify, the theorem on an annihilating sequence does not hold; in Z, evidently, there does not exist a sufficient set of discrete functionals.

One can verify that if X is some MK-lineal, then, for example, so is each of its components onto which a projection can be carried out. If, moreover, X is a continuous K-lineal with projections, then every minimal simple** l-ideal of it is an MK-lineal in which there is no unit.

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CITED LITERATURE

1. A. I. Veksler, *Izv. vyssh. uchebn. zaved.*, Mathematics, No. 1, 48 (1960).

2. A. I. Veksler, *Sibirsk. matem. zhurn.*, 3, No. 1, 7 (1962).

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3. A. I. Veksler, Dissertation, Leningrad State Pedagogical Institute named after A. I. Herzen, 1959.

4. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, 1961.

5. M. Henriksen, D. G. Johnson, *Fund. Math.*, 50, No. 1, 73 (1961).

6. D. G. Johnson, J. E. Kist, *Arch. d. Math.*, 12, No. 5, 349 (1961).

7. M. G. Krein, S. G. Krein, *Matem. sborn.*, 13, No. 1, 1 (1943).

8. H. Nakano, *Modern Spectral Theory*, Tokyo, 1950.

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* An element x of a K-lineal X is called **discrete** if from $|y| \leq |x|$, $y \in X$, it follows that $y = \alpha x$ for some real α. A regular functional is **discrete** if it is discrete as an element of the K-space of regular functionals.

** An l-ideal I is called **simple** if from $x \wedge y \in I,\ x \notin I$ it follows that $y \in I$. A simple l-ideal is called **minimal** if it contains no other simple l-ideal.

Note: Figure translations are in progress. See original paper for figures.

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