Reports of the Academy of Sciences of the USSR

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1963

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1963. Volume 149, No. 4

MECHANICS

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DESCRIPTION OF POINT SYMMETRY GROUPS BY MEANS OF TENSORS

In many problems of physics it is necessary to take into account the geometric properties of symmetry determined by point groups of orthogonal transformations. In particular, scalars, vectors, and tensors occurring in various equations that express physical laws must, in a number of cases, possess invariance properties with respect to prescribed groups of symmetry transformations.

The invariance properties specialize the form of scalar functions and of the components of the tensors under consideration. Many consequences of symmetry have been studied in detail in various applications. Existing data for various concrete examples are contained in the book by J. Nye ([1]). A more detailed treatment of the question of tensor symmetry, the properties of scalar invariants, and the construction of examples of tensors with prescribed symmetry is given in works ([2], [9]–[12]), especially in the works of A. V. Shubnikov ([4], [5]), Yu. I. Sirotin ([6]–[8]), and their collaborators.

The following proposition is valid: the geometric characteristics of the symmetry of textures and crystals (and, in other cases, those determined by the corresponding transformation groups) can be specified uniquely and completely by means of a small set of simple tensors.

The notation, the definition of the basic tensors, the basic tensors themselves, and the geometric diagrams explaining the symmetry of textures and the 32 classes of crystals are given in the appended table (see Fig. 1).

In view of the fact that only orthogonal symmetry transformations are considered, to each of the indicated bases one must adjoin the fundamental tensor g. In some cases the invariance of the fundamental tensor is a simple consequence of the invariance of the tensor basis indicated in the table (for example, for the groups Oh, Th, etc.).

It is easy to see that the choice of the corresponding sets of determining tensors can be made nonuniquely.

A proof of the formulated proposition is not difficult to give by directly checking that the requirement of invariance of the tensor basis is equivalent to specifying the system of transformation matrices that determine the given symmetry group.

The data of this table can be used to construct general formulas for the dependence of scalars and tensors on a number of other scalar and tensor quantities, taking into account their geometric symmetry properties. The corresponding formulas for tensor functions may be regarded as a generalization of the well-known Hamilton–Cayley formula, as applied to nonlinear tensor functions ([3]), to the case of several tensor arguments. A generalization of this formula to the case of dependence on several tensors of second rank is contained in works ([13], [14]) (see also ([3])).

A more detailed development of the question of tensor functions of various tensor arguments will be given by us in another paper.

Received
7 II 1963

x^1, x^2, x^3—crystallophysical Cartesian coordinates

ξ^1, ξ^2, ξ^3—arbitrary coordinates

$$a^i_j = \frac{\partial \xi^i}{\partial x^j}, \qquad \Delta = |a^i_j|$$

$$e_i = \frac{\partial \bar{r}}{\partial x^i}, \qquad \mathfrak{e}_i = \frac{\partial \bar{r}}{\partial \xi^i}$$

$$e_j = a^\alpha_j \mathfrak{e}_\alpha$$

$$g = e_1^2 + e_2^2 + e_3^2 = g^{\alpha\beta} \mathfrak{e}_\alpha \mathfrak{e}_\beta$$

$$E = e_1 e_2 e_3 - e_2 e_1 e_3 + e_2 e_3 e_1 - e_3 e_2 e_1 + e_3 e_1 e_2 - e_1 e_3 e_2 =$$

$$= \Delta \cdot (\mathfrak{e}_1 \mathfrak{e}_2 \mathfrak{e}_3 - \mathfrak{e}_2 \mathfrak{e}_1 \mathfrak{e}_3 + \mathfrak{e}_2 \mathfrak{e}_3 \mathfrak{e}_1 - \mathfrak{e}_3 \mathfrak{e}_2 \mathfrak{e}_1 + \mathfrak{e}_3 \mathfrak{e}_1 \mathfrak{e}_2 - \mathfrak{e}_1 \mathfrak{e}_3 \mathfrak{e}_2)$$

$$\Omega = e_1 e_2 - e_2 e_1 = (a^\alpha_1 a^\beta_2 - a^\alpha_2 a^\beta_1) \mathfrak{e}_\alpha \mathfrak{e}_\beta = a^\alpha_1 a^\beta_2 (\mathfrak{e}_\alpha \mathfrak{e}_\beta - \mathfrak{e}_\beta \mathfrak{e}_\alpha)$$

$$e_3^2 = a^\alpha_3 a^\beta_3 \mathfrak{e}_\alpha \mathfrak{e}_\beta$$

$$O_h = e_1^4 + e_2^4 + e_3^4 =$$

$$= (a_1^\alpha a_1^\beta a_1^\gamma a_1^\delta + a_2^\alpha a_2^\beta a_2^\gamma a_2^\delta + a_3^\alpha a_3^\beta a_3^\gamma a_3^\delta)\mathfrak{e}_\alpha \mathfrak{e}_\beta \mathfrak{e}_\gamma \mathfrak{e}_\delta$$

$$T_h = e_1^2 e_2^2 + e_2^2 e_3^2 + e_3^2 e_1^2$$

$$T_d = e_1 e_2 e_3 + e_2 e_1 e_3 + e_2 e_3 e_1 + e_3 e_2 e_1 + e_3 e_1 e_2 + e_1 e_3 e_2$$

$$D_{2h} = \lambda^{11} e_1^2 + \lambda^{22} e_2^2 + \lambda^{33} e_3^2 = \lambda^{ij} a_i^\alpha a_j^\beta \mathfrak{e}_\alpha \mathfrak{e}_\beta = d^{\alpha\beta} \mathfrak{e}_\alpha \mathfrak{e}_\beta; \quad \lambda^{11} \neq \lambda^{22} \neq \lambda^{33} \neq \lambda^{11}; \quad \lambda^{ii} \neq 0,\ d^{\alpha\beta} = d^{\beta\alpha}$$

$$C_i = D_{2h} + \omega^{ij} e_i e_j = C^{\alpha\beta} \mathfrak{e}_\alpha \mathfrak{e}_\beta, \quad \omega^{ij} = -\omega^{ji} \neq 0$$

$$D_{6h} = (e_1^3 - e_1 e_2^2 - e_2 e_1 e_2 - e_2^2 e_1)^2$$

$$D_{3h} = e_1^3 - e_1 e_2^2 - e_2 e_1 e_2 - e_2^2 e_1$$

$$D_{3d} = e_3(e_1^3 - e_1 e_2^2 - e_2 e_1 e_2 - e_2^2 e_1)$$

Textures	Cubic system	Tetragonal system	Hexagonal system	Trigonal system	Rhombic system	Monoclinic system	Triclinic system
$\infty/\infty \cdot m$[circle diagram]g	$\bar{6}/4$[cube diagram with e_1, e_2, e_3]O_h						
∞/∞[circle with three arrows]g, E	$3/4$[cube diagram]O_h, E						

Textures	Cubic system	Tetragonal system	Hexagonal system	Trigonal system	Rhombic system	Monoclinic system	Triclinic system
$m \cdot \infty : m$[axis and cylinder diagram with e_1, e_2, e_3]g, e_3^2	$\bar{3}/4$[cube diagram]g, T_d	$m \cdot 4 : m$[prism diagram]O_h, e_3^2 $\bar{4} \cdot m$[prism diagram]g, T_d, e_3^2	$m \cdot 6 : m$[hexagonal prism diagram]D_{6h}, e_3^2 $m \cdot 3 : m$[hexagonal prism diagram]D_{3h}, e_3^2	$\bar{6} \cdot m$[prism diagram]D_{3d}, e_3^2	$m \cdot 2 : m$[parallelepiped diagram with e_2, e_3]D_{2h} $\alpha = \beta = \gamma = 90^\circ$		
$\infty : 2$[axis and cylinder diagram]g, E, e_3^2	$3/2$[cube diagram]$g, E, T_d (T_h, E)$	$4 : 2$[prism diagram]O_h, E, e_3^2	$6 : 2$[hexagonal prism diagram]D_{6h}, E, e_3^2	$3 : 2$[hexagonal prism diagram]D_{3h}, E, e_3^2	$2 : 2$[parallelepiped diagram]D_{2h}, E		
$\infty : m$[axis and cylinder diagram]g, e_3^2, Ω	$\bar{6}/2$[cube diagram]T_h	$4 : m$[prism diagram]O_h, e_3^2, Ω $\bar{4}$[prism diagram]g, T_d, e_3^2, Ω	$6 : m$[hexagonal prism diagram]D_{6h}, e_3^2, Ω $3 : m$[hexagonal prism diagram]D_{3h}, e_3^2, Ω	$\bar{6}$[prism diagram]D_{3d}, e_3^2, Ω		$2 : m$[monoclinic cell diagram with e_1, e_2, e_3]D_{2h}, Ω $\alpha = \beta = 90^\circ, \gamma \neq 90^\circ$	$\bar{2}$[triclinic cell diagram with e_1, e_2, e_3]C_i $\alpha \neq \beta \neq \gamma \neq \alpha, \alpha, \beta, \gamma \neq 90^\circ$
$\infty \cdot m$[cone diagram]g, e_3		$4 \cdot m$[prism with cone diagram]O_h, e_3	$6 \cdot m$[hexagonal prism with cone diagram]D_{6h}, e_3	$3 \cdot m$[hexagonal prism with cone diagram]D_{3h}, e_3	$2 \cdot m$[parallelepiped with cone diagram]D_{2h}, e_3		
∞[cone diagram with arrows]g, E, e_3		4[prism diagram]O_h, E, e_3	6[hexagonal prism diagram]D_{6h}, E, e_3	3[hexagonal prism diagram]D_{3h}, E, e_3		2[monoclinic diagrams]D_{2h}, E, e_3 $m D_{2h}, e_1, e_2$	1[triclinic cell diagram with e_1, e_2, e_3]e_1, e_2, e_3

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Note: Figure translations are in progress. See original paper for figures.