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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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## ON THE STRUCTURE OF MULTIDIMENSIONAL SURFACES CARRYING A NET OF LINES OF CURVATURE

*(Presented by Academician P. S. Novikov, 19 XI 1962)*

1. Let  $V_m$  be an  $m$ -dimensional nonisotropic surface of the  $n$ -dimensional conformal space  $C_n$  ( $m < n$ ). As was shown in <sup>(1)</sup>, with the first-order neighborhood of a point  $A$  of this surface there is associated a quadratic form

$$\varphi^0 = g_{ij}\omega^i\omega^j,$$

which determines an angular metric on  $V_m$ , and with its second-order neighborhood—an invariant pencil of quadratic forms

$$\varphi^\alpha = a_{ij}^\alpha\omega^i\omega^j.$$

We shall call  $V_m$  a surface carrying a net of lines of curvature if at each of its points the quadratic forms  $\varphi^0$  and  $\varphi^\alpha$  can be simultaneously reduced to canonical form. Directions issuing from the point  $A$ , conjugate to one another with respect to the forms  $\varphi^0$  and  $\varphi^\alpha$ , will be called principal directions, and the lines enveloping them—the lines of curvature of the surface  $V_m$ . The principal directions are mutually orthogonal.

A hypersurface  $V_{n-1}$  of the space  $C_n$  always carries a net of lines of curvature, while for  $m < n - 1$  surfaces  $V_m$  carrying a net of lines of curvature form a special class of  $m$ -dimensional surfaces. In the present paper the structure of such surfaces is studied.

Under the Darboux transfer, a nonisotropic surface  $V_m$  of the space  $C_n$  is mapped onto a tangentially nondegenerate surface  $U_m$  lying on a second-order hypersurface  $Q_n$  of the projective space  $P_{n+1}$ . Moreover, if the surface  $V_m$  carries a net of lines of curvature, then the corresponding surface  $U_m$  carries a net of conjugate lines in the sense of our paper <sup>(2)</sup>. Therefore, in studying surfaces  $V_m$  of the space  $C_n$  carrying a net of lines of curvature, the results of <sup>(2)</sup> can be used.

2. Attach to the surface  $V_m$  a conformal frame whose point  $A_0$  coincides with the point  $A$  of the surface; the hyperspheres  $A_i$  ( $i = 1, \dots, m$ ) are orthogonal to it at the point  $A$ ; the hyperspheres  $A_\alpha$  ( $\alpha = m + 1, \dots, n$ ) are tangent to it at this point; and the point  $A_{n+1}$  serves as the second point of intersection of the hyperspheres  $A_i$  and  $A_\alpha$ . Then

$$dA_\xi = \omega_\xi^\eta A_\eta \quad (\xi, \eta = 0, 1, \dots, n + 1),$$

where the forms  $\omega_\xi^\eta$  are related by a number of relations (see (1)), one of which has the form

$$\omega_i^{n+1} = -g_{ij}\omega^j.$$

Here  $g_{ij} = (A_{iAj})$  is the tensor generating the quadratic form  $\varphi^0$ .

The surface  $V_m$  is determined by the system of Pfaff equations

$$\omega_0^\alpha = 0, \tag{1}$$

and the forms  $\omega^i = \omega_0^i$  will be linearly independent on the surface  $V_m$ . Differential prolongation of the system of equations (1) leads to a new system of equations

$$\omega_i^\alpha = \lambda_{ij}^\alpha \omega^j.$$

The coefficients  $\lambda_{ij}^\alpha$  entering here make it possible to construct a pencil of tensors

$$a_{ij}^\alpha = \lambda_{ij}^\alpha - \frac{1}{m} \lambda_{kl}^\alpha g^{kl} g_{ij},$$

by means of which an invariant pencil of quadratic forms  $\varphi^\alpha$  is introduced.

If we choose the hyperspheres  $A_i$  of the frame so that they are orthogonal to the corresponding lines of curvature of the surface  $V_m$ , then its quadratic forms  $\varphi^0$  and  $\varphi^\alpha$  take the form

$$\varphi^0 = (\omega^1)^2 + \dots + (\omega^m)^2,$$

$$\varphi^\alpha = a_1^\alpha (\omega^1)^2 + \dots + a_m^\alpha (\omega^m)^2;$$

the  $i$ -th family of lines of curvature is determined on the surface  $V_m$  by the system of equations:

$$\omega^j = 0 \quad \text{for } j \neq i.$$

Now  $g_{ij} = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol,  $\omega_i^j + \omega_j^i = 0$ , and

$$\omega_i^\alpha = \lambda_i^\alpha \omega^i, \quad \omega_i^j = \sum_k l_{ik}^j \omega^k \quad (i \neq j), \quad (2)$$

where, here and below, there is no summation over the indices  $i, j, k$  unless this is indicated by the sign  $\sum$ .

3. The second-order neighborhood of the point  $A_0$  of the surface  $V_m$  is determined by the expression

$$d^2 A_0 \equiv \sum_i (\lambda_i^\alpha A_\alpha - A_{n+1}) (\omega^i)^2 \pmod{A_0, A_i}.$$

Let us consider the hyperspheres

$$B_i = \lambda_i^\alpha A_\alpha - A_{n+1}.$$

The totality of hyperspheres passing through the point  $A_0$  and orthogonal to the hyperspheres  $A_i$  and  $B_i$  determines the first osculating sphere of the surface  $V_m$ . If by  $m_1$  we denote the number of independent hyperspheres among the  $B_i$  ( $m_1 \leq m$ ), then the dimension of this osculating sphere will be equal to  $m + m_1 - 1$ . The number  $m_1$  is equal to the number of linearly independent forms among the quadratic forms  $\varphi^0, \varphi^\alpha$ , and the number  $m_1 - 1$  is equal to the number of linearly independent forms among the quadratic forms  $\varphi^\alpha$ . We shall denote the first osculating sphere of the surface  $V_m$  by  $C_{m+m_1-1}$ .

4. The following theorems relate the structure of the surface  $V_m$  of the space  $C_n$  to the structure of its system of hyperspheres  $B_i$ .

**Theorem 1a.** *If the number  $m_1$  of linearly independent hyperspheres  $B_i$  of the surface  $V_m$  is less than  $m$ , and any subsystem of the system of hyperspheres  $B_i$  consisting of  $m - 1$  hyperspheres has rank  $m_1$ , then the surface  $V_m$  lies entirely in its first osculating sphere  $C_{m+m_1-1}$ .*

**Theorem 1b.** *If all hyperspheres  $B_i$  of the surface  $V_m$  are linearly independent, then its first osculating sphere has dimension  $2m - 1$ , and the second osculating sphere may have dimension  $2m + m_2 - 1$ , where  $0 \leq m_2 \leq m$ .*

**Theorem 1c.** *Let  $m_1 < m$ , and suppose that among the hyperspheres  $B_i$  there are  $p$  hyperspheres  $B_1, \dots, B_p$  such that, when any one of them is omitted, the system of hyperspheres  $B_i$  does not decrease in rank, and  $q = m - p$  hyperspheres  $B_{p+1}, \dots, B_m$  such that, when any one of them is omitted, the system of hyperspheres  $B_i$  decreases in rank by one. Then this surface, on the one hand,*

decomposes into a  $q$ -parameter family of surfaces  $V_p$ , each of which belongs to its own  $(p + p_1 - 1)$ -dimensional osculating sphere (here  $p_1 = m_1 - q$ ), and on the other

from the other side, into a  $p$ -parameter family of surfaces  $V_q$ , the first osculating sphere of which has dimension  $2q - 1$ , and the second osculating sphere dimension  $2q + m_2 - 1$ , where  $0 \leq m_2 \leq q$ . The surface  $V_m$  itself then has the first osculating sphere of dimension  $m + m_1 - 1$  and the second osculating sphere of dimension  $m + m_1 + m_2 - 1$ .

Theorems 1a and 1b are special cases of Theorem 1c, corresponding to  $p = m$  and  $p = 0$ .

5. We shall call a net of lines of curvature on a surface  $V_m$  of the space  $C_n$  completely holonomic if each of the Pfaff equations

$$\omega^i = 0,$$

which define the submanifolds of the net, is completely integrable. If, conversely, none of these equations and no subsystem of them consisting of fewer than  $m - 1$  equations is completely integrable, then we shall call the net of lines of curvature of the surface  $V_m$  irreducible. The condition for complete holonomicity of the net of lines of curvature of the surface  $V_m$  is that the coefficients  $l_{ik}^j$  in the expansion (2) of the forms  $\omega_i^j$  vanish for  $k \neq i$ ,  $k \neq j$ .

The following theorems relate the question of holonomicity of the net of lines of curvature on a surface  $V_m$  to the structure of its system of hyperspheres  $B_i$ .

**Theorem 2a.** *If no three hyperspheres  $B_i$  of the surface  $V_m$  belong to one pencil, then the net of lines of curvature of this surface is completely holonomic.*

**Theorem 2b.** *Only surfaces  $V_m$  of classes 1 and 0 (i.e., surfaces  $V_m$  wholly belonging to an  $(m + 1)$ -dimensional sphere, and  $m$ -dimensional spheres) can have an irreducible net of lines of curvature.*

**Theorem 2c.** *A surface  $V_m$  carrying a net of lines of curvature can decompose into a  $q$ -parameter family of  $p$ -dimensional surfaces  $V_p$  ( $p + q = m$ ), carrying an irreducible net of lines of curvature, in two cases: 1) when, among the system of hyperspheres  $B_i$  of the surface  $V_m$ ,  $p$  hyperspheres belong to one pencil, but no two of them coincide; the class of the surfaces  $V_p$  in this case is equal to one; 2) when, among the system of hyperspheres  $B_i$ ,  $p$  hyperspheres coincide with one another; the surfaces  $V_p$  in this case will be  $p$ -dimensional spheres.*

**Theorem 2d.** *If on a surface  $V_m$ , carrying a net of lines of curvature, among the system of hyperspheres  $B_i$   $p$  hyperspheres coincide with one another, then this surface is the envelope of an  $(m - p)$ -parameter family of  $m$ -dimensional spheres.*

6. Surfaces  $V_m$  carrying a holonomic net of lines of curvature are determined by the system of Pfaff equations (1), (2), in which now  $l_{ik}^j = 0$  for  $k \neq i$ ,  $k \neq j$ . Investigation of this system of equations leads to the following existence theorem:

**Theorem 3.** *Surfaces  $V_m$  carrying a holonomic net of lines of curvature, on which the systems of hyperspheres  $B_i$  satisfy the conditions of Theorems 1a and 1b, exist, and the arbitrariness of their existence is equal to  $\frac{m(m-1)}{2}$  functions of two arguments.*

The proof of the existence theorem for surfaces  $V_m$  carrying a holonomic net of lines of curvature, on which the system of hyperspheres  $B_i$  satisfies the conditions of Theorem 1c for  $p \neq m$ ,  $p \neq 0$ , is complicated by a number of technical difficulties.

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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