Reports of the Academy of Sciences of the USSR

1963

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1963. Vol. 153, No. 6

MATHEMATICS

Yu. I. Golosov, A. A. Tempelman

LIKELIHOOD RATIO FOR HYPOTHESES ON THE TREND OF CERTAIN GAUSSIAN PROCESSES

(Presented by Academician A. N. Kolmogorov on 4 VII 1963)

Let $\xi(t)$, $a \leq t \leq b$, be a real random process with zero trend, i.e. $M\xi(t) \equiv 0$ for $t \in [a, b]$. In the statistics of random processes, an important place is occupied by the problem of testing two hypotheses:

$$\text{Hypothesis } H_0: \quad \zeta(t) = \xi(t),$$

$$\text{Hypothesis } H_1: \quad \zeta(t) = \xi(t) + m(t).$$

Here $\zeta(t)$ is the observed process, and $m(t)$ is a known function. Let P_0 and P_1 be probability measures in the space $\Omega = \{\omega\}$ of functions on the segment $[a, b]$, corresponding to the processes $\xi(t)$ and $\xi(t) + m(t)$. We shall write $P_1 \ll P_0$ if the measure P_1 is absolutely continuous with respect to the measure P_0, and $P_1 \perp P_0$ if these two measures are orthogonal.

In solving the formulated problem of testing the hypotheses H_0 and H_1, two questions arise:

1) What must the function $m(t)$ be so that the condition $P_1 \ll P_0$ holds?
2) If $P_1 \ll P_0$, what is the likelihood ratio for the hypotheses H_0 and H_1, i.e. the Radon-Nikodym derivative $\dfrac{dP_1}{dP_0}(\omega)$?

For the case when $\xi(t)$ is a Wiener process, answers to these questions were given by Cameron, Martin, and Sigal (^{3,4,7}); in the case when $\xi(t)$ is an Ornstein-Uhlenbeck process (a stationary Gaussian Markov process), sufficient conditions for absolute continuity of the measure P_1 with respect to the measure P_0, and the likelihood ratio $\dfrac{dP_1}{dP_0}(\omega)$, were found by Striebel (^{9}) and Hajek (^{5}). Seguchi and Ikeda (^{8}) considered the question of testing hypotheses on a constant trend ($m(t) = \text{const}$) for Gaussian Markov processes of a more general type; however, the expression they obtained for the likelihood ratio is incorrect. A substantial generalization of the results mentioned to a broad class of Gaussian Markov processes and functions $m(t)$ satisfying certain, though rather restrictive, regularity conditions was recently obtained by Varberg (^{10}).

In the present note we indicate necessary and sufficient conditions for the absolute continuity of the measure P_1 with respect to the measure P_0 for the cases where $\xi(t)$ is a mean-square continuous Gaussian Markov process, or a Gaussian process with independent increments (having, generally speaking, fixed points of discontinuity), or a Gaussian stationary process on the entire real line; the likelihood ratios $\dfrac{dP_1}{dP_0}(\omega)$ are also found.

1. The results given below are obtained with the help of the following simple lemma, which makes it possible to reduce the problem of testing hypotheses associated with the process under consideration to the consideration of analogous hypotheses for another, "simpler" process (as which we used the Wiener process, studied in detail in the aforementioned works of Cameron, Martin, and Sigal). The possibility of such an approach to finding the ratio

of likelihood was noted in A. M. Yaglom' s survey report (^{11}); however, this device had also been used earlier (see, for example, A. V. Skorokhod (^{2})).

Lemma. Let (Ω, B) and (Ω', B') be two measurable spaces; let T be a one-to-one mapping, measurable in both directions, of Ω onto Ω'; further, let P_0 and P_1 be probability measures defined on the σ-algebra B, and let P_0' and P_1' be the measures on B' induced by the mapping T, i.e. $P_i'(A) = P_i(T^{-1}A)$, $i = 0, 1$, for any set $A \in B'$.

Then:

1) If $P_1 \ll P_0$ or $P_1 \perp P_0$, then respectively $P_1' \ll P_0'$ or $P_1' \perp P_0'$.
2) If $P_1 \ll P_0$, then

$$\frac{dP_1'}{dP_0'}(T\omega) = \frac{dP_1}{dP_0}(\omega), \qquad \omega \in \Omega,$$

almost everywhere with respect to the measure P_0.

2. Let $\xi(t)$, $a \le t \le b$, be a mean-square continuous Gaussian Markov process with correlation function $R(s,t)$. The set of points $t \in [a,b]$ at which the process $\xi(t)$ is nondegenerate, i.e. $D\xi(t) \ne 0$, is open; the values of the process $\xi(t)$ on any component interval (a_k, b_k) of this set do not depend on the values of the process outside this interval; and if $a_k < s \le t < b_k$, then $R(s,t) = \psi(s)\varphi(t)$, where $\psi(t)$ and $\varphi(t)$ are continuous functions such that the function $\psi(t)/\varphi(t)$ is nondecreasing*. Put $\psi(t) = \varphi(t) = 0$ if $D\xi(t) = 0$.

Theorem 1. $P_1 \ll P_0$ if and only if:

1) $m(t) = \varphi(t)\mu_k(\psi(t)/\varphi(t))$ for $t \in [a_k, b_k]$, where $\mu_k(\cdot)$ is a function absolutely continuous on every finite interval contained in the range of the function $\psi(t)/\varphi(t)$, $a_k \leq t \leq b_k$, $k = 1, 2, \ldots$;

2)

$$\int_a^b \left[\frac{(m(t)/\varphi(t))'}{(\psi(t)/\varphi(t))'}\right]^2 d\,\frac{\psi(t)}{\varphi(t)} < \infty^{**}. \tag{1}$$

In this case

$$\ln \frac{dP_1}{dP_0}(\omega) = -\frac{1}{2}\int_a^b \left[\frac{(m(t)/\varphi(t))'}{(\psi(t)/\varphi(t))'}\right]^2 d\,\frac{\psi(t)}{\varphi(t)} +$$

$$+\int_a^b \left[\frac{(m(t)/\varphi(t))'}{(\psi(t)/\varphi(t))'}\right]^2 d\,\frac{\xi(t)}{\varphi(t)} - \frac{1}{2}\,\frac{m^2(a)}{\psi(a)\varphi(a)} + \frac{m(a)\xi(a)}{\psi(a)\varphi(a)}. \tag{2}$$

The proof of the theorem follows from the fact that, if the function $\psi(t)/\varphi(t)$, $a_k \leq t \leq b_k$, is strictly increasing, then the transformation $w(u) = \xi(g(u))/\varphi(g(u))$, where $g^{-1}(t) = \psi(t)/\varphi(t)$, carries the process $\xi(t)$, $a_k \leq t \leq b_k$, into a Wiener process on the interval $[\psi(a_k)/\varphi(a_k), \psi(b_k)/\varphi(b_k)]$.

Remark. From conditions 1)-2) it follows that if $D\xi(t_0) = 0$, then $m(t) = O(\sqrt{D\xi(t)})$ as $t \to t_0$; if $D\xi(t_0) = 0$ and, moreover,

$$\lim_{t \to t_0 - 0} \psi(t)/\varphi(t) = 0$$

or

$$\lim_{t \to t_0 + 0} \psi(t)/\varphi(t) = \infty,$$

then $m(t) = o(\sqrt{D\xi(t)})$, respectively, as $t \to t_0 - 0$ or $t \to t_0 + 0$.

3. Let $\xi(t)$, $a \leq t \leq b$, be a Gaussian process with independent increments, with $\xi(a) = 0$; let $t_1, t_2, \ldots$ be the set of discontinuity points (jumps) of this process; put $F(t) = D\xi(t)$.

Theorem 2. $P_1 \ll P_0$ if and only if:

1) $m(t) = \mu[F(t)]$, where the function $\mu(t)$ is absolutely continuous on any—

---

[*] This fact follows from Theorem 8.1 of Chapter 5 of the book (1).

[**] We put $0/0 = 0$, $1/0 = \infty$. The integrals in formulas (1), (2) should be understood as sums of integrals over intervals of monotonicity of the function $\psi(t)/\varphi(t)$.

on every finite segment $[c, d] \subset [0, F(b)]$;

2)

$$\int_a^b \left[\frac{m'(t)}{F'(t)}\right]^2 dF_c(t) + \sum_{i=1}^{\infty} \frac{[\Delta_- m(t_i)]^2}{\Delta_- F(t_i)} + \sum_{i=1}^{\infty} \frac{[\Delta_+ m(t_i)]^2}{\Delta_+ F(t_i)} < \infty^*.$$

Then

$$\begin{aligned}\ln \frac{dP_1}{dP_0}(\omega) = & -\frac{1}{2}\int_a^b \left[\frac{m'(t)}{F'(t)}\right]^2 dF_c(t) - \frac{1}{2}\sum_{i=1}^{\infty} \frac{[\Delta_- m(t_i)]^2}{\Delta_- F(t_i)} \\ & -\frac{1}{2}\sum_{i=1}^{\infty} \frac{[\Delta_+ m(t_i)]^2}{\Delta_+ F(t_i)} + \int_a^b \frac{m'(t)}{F'(t)}\, d\xi_c(t) \\ & + \sum_{i=1}^{\infty} \frac{\Delta_- m(t_i)\Delta_-\xi(t_i)}{\Delta_- F(t_i)} + \sum_{i=1}^{\infty} \frac{\Delta_+ m(t_i)\Delta_+\xi(t_i)}{\Delta_+ F(t_i)}.\end{aligned}$$

The proof of this theorem can be obtained by combining the assertion of Theorem 1, applied to the process $\xi_c(t)$, with Kakutani' s result (6).

4. With the aid of the lemma of § 1 it is easy to prove the following result (see also (12)):

Theorem 3. Let $\xi(t)$, $-\infty < t < \infty$, be a Gaussian stationary process continuous in mean square; let $F(d\lambda)$ be its spectral measure and $Z(d\lambda)$ its random spectral measure.

$P_1 \ll P_0$ if and only if:

1)

$$m(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, G(d\lambda),$$

where $G(d\lambda)$ is a complex measure;

2) $G(d\lambda) \ll F(d\lambda)$;

3) $\dfrac{dG}{dF}(\lambda) \in L_2(F)$.

Then

$$\ln \frac{dP_1}{dP_0}(\omega) = -\frac{1}{2}\int_{-\infty}^{+\infty} \left|\frac{dG}{dF}(\lambda)\right|^2 (d\lambda) + \int_{-\infty}^{+\infty} \frac{dG}{dF}(\lambda)\, Z(d\lambda).$$

In the proof, the result established in Theorem 2 is used.

Remark. From conditions 1) and 2) of Theorem 3 it follows that the measure $G(d\lambda)$ has bounded variation and the function $m(t)$ is uniformly continuous.

The authors express their gratitude for the attention of A. M. Yaglom, who suggested to them the topic of the present work.

Institute of Physics and Mathematics
Academy of Sciences of the Lithuanian SSR

Received
4 VII 1963

CITED LITERATURE

[1] J. L. **Doob**, *Stochastic Processes*, Moscow, 1956.
[2] A. V. **Skorokhod**, *Studies in the Theory of Random Processes*, Kiev, 1961.
[3] R. C. **Cameron**, W. T. **Martin**, Ann. Math., **45**, 386 (1944).
[4] R. C. **Cameron**, Duke Math. J., **21**, 623 (1954).
[5] J. **Hajek**, Trans. 2-d Prague Conf. Inform. Theory Stat. Decis. Funct., Random Processes, Prague, 1950.
[6] S. **Kakutani**, Ann. Math., **49**, 214 (1948).
[7] I. E. **Segal**, Trans. Am. Math. Soc., **88**, 12 (1958).
[8] T. **Seguchi**, N. **Ikeda**, Mem. Fac. Sci. Kyusyu Univ., Ser. A8, No. 11, 187 (1954).
[9] Ch. **Stribel**, Ann. Math. Stat., **30**, 559 (1959).
[10] D. E. **Varberg**, Proc. Am. Math. Soc., **13**, 799 (1962).
[11] A. M. **Yaglom**, *Time Series Analysis*, Ch. 22, N. Y.–London, 1963.
[12] E. **Parzen**, Proc. IV Berkeley Sympos., 1960, **1**, 1961, p. 469.

[*] We denote: $\Delta_+ f(a) = \lim_{t\to a+0} f(t) - f(a), \quad \Delta_- f(a) = f(a) - \lim_{t\to a-0} f(t),$

$$f_d(t) = \sum_{t_i \le t} \Delta_- f(t_i) + \sum_{t_i < t} \Delta_+ f(t_i), \quad f_c(t) = f(t) - f_d(t).$$

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.