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Abstract

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ON THE SOLUTION BY THE FOURIER METHOD OF A ONE-DIMENSIONAL MIXED PROBLEM FOR QUASILINEAR HYPERBOLIC EQUATIONS OF SECOND ORDER

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The present work is a development and continuation of the works ([1a,b,c]), in which the existence and uniqueness of a generalized solution, of a solution almost everywhere, and of a classical solution of the following one-dimensional mixed problem were studied:

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = \lambda F\left[t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right],$$

$$u(t,0) = u(t,l) = 0, \tag{A}$$

$$u(0,x) = \varphi(x), \qquad u'_t(0,x) = \psi(x),$$

where $0 \le t \le T < \infty$; $0 \le x \le l < \infty$; $a > 0$; λ is a parameter; F, φ, ψ are given functions.

In the present paper we establish theorems on the existence and uniqueness of a generalized solution, of a solution almost everywhere, and of k-times ($k \ge 2$) continuously differentiable solutions of problem (A). We formulate the main results obtained.

§ 1. Generalized solution of problem (A).

Theorem 1. Suppose:

1. The function $\varphi(x)$ is continuous on the interval $[0, l]$, $\varphi'(x) \in L_2[0, l]$, $\varphi(0) = \varphi(l) = 0$; $\psi(x) \in L_2[0, l]$.
2. $F[t, x, 0, 0, 0] \in L_2(D)$, where $D = (0 \le t \le T) \times (0 \le x \le l)$.
3. The function $F[t, x, u, v, w]$, defined in the domain

$$D \times (-R < u < R) \times (-\infty < v, w < \infty),$$

is measurable with respect to (t,x) for fixed (u,v,w) and, for almost all $(t,x)\in D$, satisfies the condition

$$|F[t,x,u,v,w]-F[t,x,\tilde{u},\tilde{v},\widetilde{w}]|\le a(t,x)|u-\tilde{u}|+$$
$$+\,b(t)\,[\,|v-\tilde{v}|+|w-\widetilde{w}|\,],$$

where $a(t,x)\in L_2(D)$, $b(t)\in L_2[0,T]$,

$$R>\frac{V^2 l\,(l^2+a^2\pi^2)}{al}\left[a^2\|\varphi'(x)\|^2_{L_2[0,l]}+\|\psi(x)\|^2_{L_2[0,l]}+\right.$$
$$\left.+\lambda^2T\|F[t,x,0,0,0]\|^2_{L_2(D)}\right]^{1/2}\times$$
$$\times\sum_{s=0}^{\infty}\left\{\frac{\left(\frac{\lambda^2T(l^2+a^2\pi^2)}{a^2\pi^2l^2}\left[\pi^2l\|a(t,x)\|^2_{L_2(D)}+3(l^2+\pi^2)\|b(t)\|^2_{L_2[0,T]}\right]\right)^s}{s!}\right\}^{1/2}.$$

Then problem (A) has a unique generalized solution satisfying the condition

$$\max_{(t,x)\in D}|u(t,x)|<R \tag{1}$$

and depending continuously on the initial functions $\varphi(x),\psi(x)$ and the parameter λ in the sense that a small change in the numbers $\|\varphi'(x)\|_{L_2[0,l]}$, $\|\psi(x)\|_{L_2[0,l]}$ and ···

λ there corresponds a change of this solution that is small in the norm of the space $B^{\infty,2}_{1,0}$ (([4r]), p. 17).

Theorem 2. Let:

1. The first and second conditions of Theorem 1 be satisfied.
2. The third condition of Theorem 1 be satisfied, where

$$R>N\equiv\frac{1}{a}\sqrt{\frac{2}{3l}\,(l+a\pi)\left(a^2\|\varphi'(x)\|^2_{L_2[0,l]}+\|\psi(x)\|^2_{L_2[0,l]}\right)}.$$

3. The inequality

$$\frac{2|\lambda|(l+a\pi)\sqrt{3T}}{3al\pi}\left\{l\pi^2\|F[t,x,0,0,0]\|^2_{L_2(D)}+[l\pi^2]\|a(t,x)\|^2_{L_2(D)}+3(l^2+\pi^2)\|b(t)\|^2_{L_2[0,T]}R^2\right\}^{1/2}<R-N.$$

Then problem (A) has a unique generalized solution satisfying condition (1) and depending continuously on the initial functions $\varphi(x)$, $\psi(x)$ and on the parameter λ in the sense indicated in Theorem 1.

Now, along with problem (A), let us consider the problem:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \tilde{\lambda}\Phi\left[t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right],$$

$$u(t,0) = u(t,l) = 0,$$

$$u(0,x) = \widetilde{\varphi}(x), \qquad u'_t(0,x) = \widetilde{\psi}(x). \tag{$\widetilde{\text{A}}$}$$

Theorem 3. Let:

1. The functions $\varphi(x)$, $\widetilde{\varphi}(x)$, $\psi(x)$, $\widetilde{\psi}(x)$ satisfy, respectively, condition 1 of Theorem 1.
2. The functions $F[t,x,u,v,w]$, $\Phi[t,x,u,v,w]$, defined in the domain $D \times (-\infty < u, v, w < \infty)$, be measurable in (t,x) for all fixed (u,v,w) and, for almost all $(t,x) \in D$, satisfy the condition

$$|F[t,x,u,v,w] - F[t,x,\tilde{u},\tilde{v},\widetilde{w}]| \le a(t,x)|u-\tilde{u}| + b(t)\big[|v-\tilde{v}| + |w-\widetilde{w}|\big],$$

$$|\Phi[t,x,u,v,w] - \Phi[t,x,\tilde{u},\tilde{v},\widetilde{w}]| \le \tilde{a}(t,x)|u-\tilde{u}| + \tilde{b}(t)\big[|v-\tilde{v}| + |w-\widetilde{w}|\big],$$

where $a(t,x), \tilde{a}(t,x) \in L_2(D)$, $b(t), \tilde{b}(t) \in L_2[0,T]$.

3. $F[t,x,0,0,0]$, $\Phi[t,x,0,0,0] \in L_2(D)$.
4. In the domain $D \times (-R \le u \le R) \times (-\infty < v, w < \infty)$ one has

$$|F[t,x,u,v,w] - \Phi[t,x,u,v,w]| \le c(t,x),$$

where $c(t,x) \in L_2(D)$,

$$R^2 = \frac{(l^2 + a^2\pi^2)}{a^2 l}\left[a^2\|\varphi'(x)\|^2_{L_2[0,l]} + \|\psi(x)\|^2_{L_2[0,l]} + 4\lambda^2 T\|F[t,x,0,0,0]\|^2_{L_2(D)}\right] \times$$

$$\times \exp\left\{\frac{4\lambda^2 T(l^2 + a^2\pi^2)}{a^2\pi^2 l^2}\left[l\pi^2\|a(t,x)\|^2_{L_2(D)} + 3(l^2+\pi^2)\|b(t)\|^2_{L_2[0,T]}\right]\right\}.$$

Then for any $\varepsilon > 0$ one can specify such a $\delta(\varepsilon) > 0$ that

$$\|u(t,x)-\tilde{u}(t,x)\|_{B^{\infty,2}_{1,0}}<\varepsilon,$$

provided only that each of the quantities

$$\|\varphi'(x)-\widetilde{\varphi}'(x)\|_{L_2[0,l]},\quad \|\psi(x)-\widetilde{\psi}(x)\|_{L_2[0,l]},\quad \|c(t,x)\|_{L_2(D)},\quad |\lambda-\tilde{\lambda}|$$

is less than δ, where $u(t,x)$, $\tilde{u}(t,x)$ are, respectively, the generalized solutions of problems (A), ($\widetilde{\text{A}}$).

Theorem 4. If condition 3 of Theorem 1 is satisfied, where R is any positive number, then problem (A) has no more than one generalized solution satisfying condition (1).

§ 2. Solution almost everywhere of problem (A).

Theorem 5. Suppose:

1. The function $\varphi(x)$ is continuously differentiable on the interval $[0,l]$, $\varphi''(x)\in L_2[0,l]$, $\varphi(0)=\varphi(l)=0$.
2. The function $\psi(x)$ is continuous on the interval $[0,l]$, $\psi'(x)\in L_2[0,l]$, $\psi(0)=\psi(l)=0$.
3. The function $F[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]$, continuous in the domain $Q=D\times(-R<\xi_3,\xi_4,\xi_5<R)$ with respect to the totality of its arguments, has partial derivatives F'_{ξ_i} $(i=2,\dots,5)$, defined in this same domain and measurable with respect to (ξ_1,ξ_2) for fixed (ξ_3,ξ_4,ξ_5), where

$$R^2>\frac{2(l+a\pi)^2(\max\{l,\pi\})^2}{3a^2\pi^2 l}\left(a^2\|\varphi''(x)\|^2_{L_2[0,l]}+\|\psi'(x)\|^2_{L_2[0,l]}\right).$$

4. Almost for all $(\xi_1,\xi_2)\in D$ one has

$$\left|F'_{\xi_i}[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]-F'_{\xi_i}[\xi_1,\xi_2,\tilde{\xi}_3,\tilde{\xi}_4,\tilde{\xi}_5]\right|\le a(\xi_1,\xi_2)\sum_{s=3}^{5}|\xi_s-\tilde{\xi}_s|,$$

$$\left|F'_{\xi_i}[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]-F'_{\xi_j}[\xi_1,\xi_2,\tilde{\xi}_3,\tilde{\xi}_4,\tilde{\xi}_5]\right|\le b(\xi_1)\sum_{s=3}^{5}|\xi_s-\tilde{\xi}_s|,$$

where $a(\xi_1,\xi_2)\in L_2(D)$, $b(\xi_1)\in L_2[0,T]$, $i=2,3$, $j=4,5$.

5. $F[\xi_1,0,0,0,\xi_5]=F[\xi_1,l,0,0,\xi_5]=0$, $F'_{\xi_i}[\xi_1,\xi_2,0,0,0]\in L_2(D)$,

$$\sup_{0<x<l}|F'_{\xi_j}[t,x,0,0,0]|=g_j(t)\in L_2[0,T],$$

where $i=2,3$, $j=4,5$.

Then problem (A) has a unique solution almost everywhere, satisfying in D the condition

$$-R < u(t,x),\ u'_t(t,x),\ u'_x(t,x) < R, \tag{2}$$

under each of the following conditions: a) one of the numbers $T, |\lambda|$ is fixed, and the other is sufficiently small; b) λ, T are fixed; $\|\varphi''(x)\|_{L_2[0,l]}$, $\|\psi'(x)\|_{L_2[0,l]}$, $\|F'_x[t,x,0,0,0]\|_{L_2(D)}$ are sufficiently small.

Theorem 6. If the function $F[t,x,u,v,w]$, defined in the domain $D \times (-R_1 < u < R_1) \times (-R_2 < v < R_2) \times (-R_3 < w < R_3)$, almost for all $(t,x) \in D$ satisfies the condition

$$|f[t,x,u,v,w] - F[t,x,\tilde{u},\tilde{v},\widetilde{w}]| \leq a(t,x)|u-\tilde{u}| +$$

$$+b(t)\left[|v-\tilde{v}| + |w-\widetilde{w}|\right],$$

where $a(t,x) \in L_2(D)$, $b(t) \in L_2[0,T]$, then problem (A) has at most one solution almost everywhere satisfying in D the conditions

$$|u(t,x)| < R_1, \qquad |u'_t(t,x)| < R_2, \qquad |u'_x(t,x)| < R_3.$$

§ 3. k-times ($k \geq 2$) continuously differentiable in D solutions of problem (A).

Theorem 7. Suppose:

1. The function $\varphi(x)$ is k-times continuously differentiable on the interval $[0,l]$, $\varphi^{k+1}(x) \in L_2[0,l]$, $\varphi^{(2m)}(0) = \varphi^{(2m)}(l) = 0$, where $m = 0, \dots, [k/2]$.
2. The function $\psi(x)$ is $k-1$ times continuously differentiable on the interval $[0,l]$, $\psi^{(k)}(x) \in L_2[0,l]$, $\psi^{(2m)}(0) = \psi^{(2m)}(l) = 0$, where $m = 0, \dots, [(k-1)/2]$.
3. The function $F[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]$, defined in the domain $Q = D \times (-R < \xi_3,\xi_4,\xi_5 < R)$, has partial derivatives $\partial^k F/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}$,

defined in the same domain, where

$$R^2 > \frac{2l^{2k-3}(l+\alpha\pi)^2(\max\{1,T\})^2}{45\,\alpha^2\pi^{2(k-1)}}\left[a^2\|\varphi^{(k+1)}(x)\|^2_{L_2[0,l]} + \|\psi^{(k)}(x)\|^2_{L_2[0,l]}\right].$$

4. The functions
   $\partial^s F/\partial\xi_1^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}$ $(s = 0, \dots, k-2)$,
   $\partial^s F/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}$ $(s = 1, \dots, k-1)$ are continuous jointly in their arguments in the domain Q.

5. The functions
$\partial^k F/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}$ are measurable in (ξ_1,ξ_2) for fixed (ξ_3,ξ_4,ξ_5) and, for almost all $(\xi_1,\xi_2)\in D$, satisfy the condition

$$\left|\frac{\partial^k F[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]}{\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}}-\frac{\partial^k F[\xi_1,\xi_2,\tilde{\xi}_3,\tilde{\xi}_4,\tilde{\xi}_5]}{\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}}\right|\le a(\xi_1,\xi_2)\sum_{s=3}^{5}|\xi_s-\tilde{\xi}_s|,$$

where $a(\xi_1,\xi_2)\in L_2(D)$.

6.

$$\partial^k F[\xi_1,\xi_2,0,0,0]/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}\in L_2(D),$$

$$\partial^{2s} F[\xi_1,0,0,0,\xi_5]/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}=\partial^{2s} F[\xi_1,l,0,0,\xi_5]/\partial\xi_2^{k_1}\partial\xi_3^{k_2}\partial\xi_4^{k_3}\partial\xi_5^{k_4}\equiv 0,$$

where $s=0,\dots,[(k-1)/2]$.

Then, under each of the following conditions, problem (A) has a unique solution in D, continuously differentiable k times, satisfying condition (2) in D: a) one of the numbers T, $|\lambda|$ is fixed, and the other is sufficiently small; b) λ, T are fixed, while
$\|\varphi^{(k+1)}(x)\|_{L_2[0,l]}$, $\|\psi^{(k)}(x)\|_{L_2[0,l]}$,
$\left\|\partial^k F[\xi_1,\xi_2,0,0,0]/\partial\xi_2^k\right\|_{L_2(D)}$ are sufficiently small.

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Note: Figure translations are in progress. See original paper for figures.