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Abstract

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SINGULARITIES OF FUNDAMENTAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Consider a solution of a linear partial differential equation with constant real coefficients in three independent variables:

$$\mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathcal{E}(x,y,z) = \delta(x,y,z), \qquad (x,y,z) \in R^3,$$

where $\mathcal{L}(\alpha,\beta,\gamma)$ is a homogeneous polynomial of degree $m > 4$. We shall call the function $\mathcal{E}(x,y,z)$ a **fundamental solution** of the operator $\mathcal{L}$. It is clear that $\mathcal{E}(x,y,z)$ is a homogeneous function.

In the case when the algebraic curve $\mathcal{L}(\alpha,\beta,1) = 0$ has no real singular points, the construction of the function $\mathcal{E}(x,y,z)$ was given by Zeilon ([1]). In the case when the algebraic curve $\mathcal{L}(\alpha,\beta,1) = 0$ has real singular points, the construction of the function $\mathcal{E}(x,y,z)$ was given by the author in ([4]).

The function $\mathcal{E}(x,y,z)$ is locally analytic at all points of the space R^3, except for the points of the characteristic cone $K(x,y,z) = 0$ of the operator $\mathcal{L}$. This theorem, in the case when the algebraic curve $\mathcal{L}(\alpha,\beta,1) = 0$ has no real singular points, was proved by V. A. Borovikov ([2]). The proof in the case when the algebraic curve $\mathcal{L}(\alpha,\beta,1) = 0$ has real singular points is due to the author ([3]).

V. A. Borovikov investigated the behavior of the function $\mathcal{E}(x,y,z)$ in a neighborhood of an ordinary point of the characteristic cone $K(x,y,z) = 0$ of the operator $\mathcal{L}$ ([2]). In the present work we describe the behavior of the function $\mathcal{E}(x,y,z)$ in a neighborhood of certain singular points of the characteristic cone. The method of investigation is the usual method of the theory of integrals on algebraic curves, the method of local expansions ([5,6]).

By virtue of the homogeneity of the function $\mathcal{E}(x,y,z)$, its study in a neighborhood of points of the cone $K(x,y,z) = 0$ can be reduced to the study of the function $E(x,y) = \mathcal{E}(x,y,1)$ in a neighborhood of points of the algebraic curve $\Gamma(x,y) = K(x,y,1) = 0$. It can be shown that the algebraic curve $\Gamma(x,y) = 0$ is dual ([4]) to the algebraic curve $Q(\alpha,\beta) = \mathcal{L}(\alpha,\beta,1) = 0$. In the present work we use the terminology of the theory of algebraic curves adopted in the monograph of R. Walker ([4]).

We shall assume that the algebraic curve $Q(\alpha, \beta) = 0$ is irreducible and has the simplest real singular points (double points), and we shall describe the behavior of the function $E(x, y)$ in a neighborhood of the singular points of the algebraic curve $\Gamma(x, y) = 0$. In this case the problem of investigating the fundamental solution of the operator $\mathcal{L}$ at all points of the space R^3 is completely solved. The same method of investigating the function $E(x, y)$ can also be used in the case when the algebraic curve $Q(\alpha, \beta) = 0$ has more complicated singular points; however, here a painstaking study of the structure of the dual curve $\Gamma(x, y) = 0$ in a neighborhood of the corresponding points is required, and we have not undertaken this.

The function under consideration $E(x, y)$ has the form

$$E(x, y) = \operatorname{Re} \frac{i}{4\pi\Gamma(m-2)} \sum_{\nu} \int_{\beta_\mu(x,y)}^{\beta_\nu(x,y)} \frac{(\alpha_\nu x + \beta y + 1)^{m-3} \alpha\beta}{Q_\alpha(\alpha_\nu, \beta)};$$

the integration in each term of the sum is carried out along the algebraic curve $Q(\alpha, \beta) = 0$; the upper limits of the integrals in the sum are points of this curve, whose coordinates $\beta_\nu(x, y)$ are complex roots with positive imaginary part for $x > 0$ (with negative imaginary part for $x < 0$) of the algebraic equation

$$Q\left(-\frac{1}{x}(1 + \beta y), \beta\right) = 0;$$

the summation in each region of the x, y-plane is performed over all such roots of this equation; $\beta_\mu(x, y)$ is the root of the same equation complex-conjugate to the root $\beta_\nu(x, y)$ in the region where these roots are complex; in the region of the x, y-plane in which the roots $\beta_\nu(x, y)$ and $\beta_\mu(x, y)$ are real, the value of the corresponding integral of the sum is determined by continuity.

We shall give the expansion of the function $E(x, y)$ in a neighborhood of a singular point of the curve $\Gamma(x, y) = 0$ corresponding to an ordinary point of the curve $Q(\alpha, \beta) = 0$ with zero curvature; in a neighborhood of an isolated singular point of the curve $\Gamma(x, y) = 0$; in a neighborhood of an inflection point of the curve $\Gamma(x, y) = 0$, corresponding to a cusp of the first kind of the curve $Q(\alpha, \beta) = 0$; in a neighborhood of the points of a real double tangent to the curve $\Gamma(x, y) = 0$, corresponding to a point of self-intersection of the curve $Q(\alpha, \beta) = 0$; in a neighborhood of a point of self-tangency of the curve $\Gamma(x, y) = 0$, corresponding to the same kind of point of the curve $Q(\alpha, \beta) = 0$.

The expansion of the function $E(x, y)$ in a neighborhood of the point under consideration will in all cases have, in specially chosen local coordinates u, v, analytically depending on the variables x, y, analytically invertible and such that $\partial(u, v)/\partial(x, y) = 1$, the form

$$E(x, y) = E'(u, v) = \Phi(u, v) + \Psi(u, v),$$

where $\Psi(u, v)$ is an analytic function; the expansion of the function $\Phi(u, v)$ will be specified in each case.

1. Let the point $(1, 0)$ of the algebraic curve $Q(\alpha, \beta) = 0$ be an ordinary point with zero curvature, and let the function $Q(\alpha, \beta)$ in a neighborhood of this point have the expansion

$$Q(\alpha, \beta) = (\alpha - 1) + \frac{(n-1)^{n-1}}{n^n}\beta^n + \cdots.$$

Then the corresponding point $(-1, 0)$ of the dual curve $\Gamma(x, y) = 0$ will be a singular point, in a neighborhood of which one can introduce such local coordinates u, v that the function $\Gamma(x, y)$ in these coordinates takes the form:

$$\Gamma(x, y) = \Gamma'(u, v) = [u^{n-1} + (-1)^{n-1}v^n]\Lambda(u, v),$$

where $\Lambda(u, v)$ is an analytic function satisfying the condition $\Lambda(0, 0) = 1$.

The function $\Phi(u, v)$ in a neighborhood of the point $(0, 0)$ has the expansion

$$\Phi(u, v) = \sum_{i=1}^{i=[n/2]} \Phi_i(u, v),$$

where

$$\Phi_1(u, v) = \begin{cases} |u|^{m-3+1/n} f_k(\varphi_1(k)) + \chi(k, u), & \text{for } k < 1, \\ 0, & \text{for } k > 1; \end{cases}$$

$$\Phi_i(u, v) = |u|^{m-3+1/n} f_k(\varphi_i(k)) + \chi(k, u) \quad \text{for all } k, \quad i = 2, \ldots, \left[\frac{n}{2}\right];$$

$$f_k(\varphi_i(k)) = \frac{i}{4\pi\Gamma(m-2)} \int_{\varphi(k)}^{\varphi(k)} \left[1 - zk^{1/n} + \frac{(n-1)^{n-1}}{n^n} z^n\right]^{m-3} dz;$$

$$k = (-1)^n v^n / u^{n-1};$$

the summation is carried out over the complex roots of the equation

$$\frac{(n-1)^{n-1}}{n^n}\varphi^n - k^{1/n}\varphi + 1 = 0,$$

satisfying the condition $\operatorname{Im}\varphi(k)u^{1/n} > 0$ (for $k > 1$ two roots of this equation become real); $\chi(k,u)$ is a function containing, for every fixed k, terms of higher order with respect to u.

2. Let the algebraic curve $\Gamma(x,y) = 0$ have an isolated singular point with coordinates $(-1,0)$, and let the function $\Gamma(x,y)$ in a neighborhood of this point have the expansion

$$\Gamma(x,y) = (x+1)^2 + y^2 + a(x+1)^3 + b(x+1)y^2 + \cdots$$

It can be shown that the function $E(x,y)$ is analytic in a neighborhood of the point $(-1,0)$.

3. Let the algebraic curve $Q(\alpha,\beta) = 0$ have a cusp of the first kind with coordinates $(1,0)$, and let the function $Q(\alpha,\beta)$ in a neighborhood of this point have the expansion

$$Q(\alpha,\beta) = (\alpha-1)^2 - \beta^3 + \cdots$$

The corresponding part of the dual curve $\Gamma(x,y) = 0$ consists of two components: the branch of the curve having a point of inflection, whose equation in the corresponding local coordinates is

$$u + \frac{4}{27}v^3 = 0,$$

and the tangent line to this branch with equation $u = 0$.

The function $\Phi(u,v)$ in a neighborhood of the point $(0,0)$ has the form

$$\Phi(u,v) = \begin{cases} u^{m-1/6} f_k(\varphi(k)) + \chi(k,u), & \text{for } -\dfrac{4}{27} < k < \infty, \\ 0, & \text{for } -\infty < k < -\dfrac{4}{27}, \end{cases}$$

where

$$f_k(\varphi(k)) = \frac{i}{8\pi\Gamma(m-2)} \int_{\varphi(k)}^{\overline{\varphi(k)}} \frac{(1 + zk^{1/3} - z^{1/2})^{m-3} - (1 + zk^{1/3} + z^{1/2})^{m-3}}{z^{3/2}}\, dz;$$

$$k = v^3/u;$$

$\varphi(k)$ is the complex root with positive imaginary part of the cubic equation

$$\varphi^3 - k^{2/3}\varphi^2 - 2k^{1/3}\varphi - 1 = 0 \qquad \left(-\frac{4}{27} < k < \infty\right)$$

(for $-\infty < k < -\frac{4}{27}$ the roots of this equation are real); $\chi(k,u)$ is a function containing, for every fixed k, terms of higher order with respect to u.

4. Let the algebraic curve $Q(\alpha,\beta) = 0$ have a self-intersection point with coordinates $(1,0)$, and let the function $Q(\alpha,\beta)$ in a neighborhood of the point $(1,0)$ have the expansion

$$Q(\alpha,\beta) = (\alpha-1)^2 - \beta^2 + a(\alpha-1)^3 + b(\alpha-1)\beta^2 + \cdots, \qquad a > 0,\ b > 0.$$

The corresponding part of the dual curve $\Gamma(x,y) = 0$ consists of three components: two branches of the curve, having the following local equations in the corresponding local coordinates:

$$u = -\frac{1}{4l}v_i^2, \qquad l = \frac{a+b}{2}, \qquad i = 1,2$$

(the first branch passes through the point $(-1,+1)$, the second through the point $(-1,-1)$), and their common tangent with equation $u = 0$ $(u = x+1)$.

The function $\Phi(u,v_i)$ in a neighborhood of the point $(0,0)$ has the expansion

$$\Phi(u,v_i) = \begin{cases} u^{m-1/2} f_k(\varphi(k)) + \chi_i(k,u), & \text{for } k+4l<0, \\ 0, & \text{for } k+4l>0,\ v_1>0\ (v_2<0), \\ \sigma(u), & \text{for } k+4l>0,\ v_1<0\ (v_2>0). \end{cases}$$

where

$$f_k(\varphi(k)) = \frac{-i}{4\pi\Gamma(m-3)} \int_{\overline{\varphi}(k)}^{\varphi(k)} (1 + zk^{1/2} - lz^2)^{m-4}\,dz;$$

$k = v_i^2/u$; $\varphi(k)$ is the complex root with positive imaginary part of the quadratic equation

$$l\varphi^2 - \varphi k^{1/2} - 1 = 0 \qquad (k+4l<0)$$

(for $k+4l>0$ the roots of this equation are real); $\chi_i(k,u)$ is a function which, for every fixed k, contains terms of higher order relative to u; $\sigma(u)$ is an analytic function.

The expansion of the function $E(x,y)$ along the normal to the double tangent has the form

$$E(x,y) = \begin{cases} -\dfrac{(x+1)^{m-3}}{2\pi\Gamma(m-2)} + o\big[(x+1)^{m-2}\big], & \text{for } |y|<1,\ u>0, \\ 0, & \text{for } |y|>1,\ u>0, \\ 0, & \text{for any } y,\ u<0. \end{cases}$$

5. If the algebraic curve $\Gamma(x,y)=0$ has an isolated double tangent, then one can show that the function $E(x,y)$ is analytic in a neighborhood of this line.
6. Let the algebraic curve $Q(\alpha,\beta)=0$ have a point of self-tangency with coordinates $(1,0)$, and let the function $Q(\alpha,\beta)$ in a neighborhood of this point have the expansion

$$Q(\alpha,\beta) = [(\alpha-1)-b_1\beta^2]\,[(\alpha-1)-b_2\beta^2] + \dots, \qquad b_2<b_1<0.$$

The corresponding point $(1,0)$ of the dual curve $\Gamma(x,y)=0$ is also a point of self-tangency, in a neighborhood of which one can introduce local coordinates u,v, in which the function $\Gamma(x,y)$ takes the form

$$\Gamma(x,y) = \Gamma'(u,v) = \left(u + \frac{1}{4b_1}v^2\right)\left(u + \frac{1}{4b_2}\chi(v)\right)\Lambda(u,v),$$

where $\chi(v)$ is an analytic function satisfying the conditions $\chi(0)=\chi'(0)=0,\ \chi''(0)=2$; $\Lambda(u,v)$ is an analytic function satisfying the condition $\Lambda(0,0)=1$.

In this case, in a neighborhood of the point $(0,0)$, the function $\Phi(u,v)$ has the expansion

$$\Phi(u,v) = \begin{cases} u^{m-7/2}\,[f_{1k}(\varphi_1(k)) + f_{2k}(\varphi_2(k))] + \chi_1(k,u), & \text{for } k+4b_1<0, \\ u^{m-7/2} f_{2k}(\varphi_2(k)) + \chi_2(k,u), & \text{for } k+4b_2<0, \\ 0, & \text{for } k+4b_2>0, \end{cases}$$

where

$$f_{jk}(\varphi_j(k)) = \frac{i}{4\pi\Gamma(m-2)} \int_{\overline{\varphi}j(k)}^{\varphi_j(k)} \frac{(1+zk^{1/2}-b_1z^2)^{m-3} - (1+zk^{1/2}-b_2z^2)^{m-3}}{(b_1-b_2)z^2}\,dz;$$

$k = v^2/u$; $\varphi_j(k)$ is the complex root with positive imaginary part of the quadratic equation

$$b_j\varphi^2 - \varphi k^{1/2} - 1 = 0 \qquad (k + 4b_j < 0;\ j = 1, 2)$$

(for $k+4b_j > 0$ the roots of this equation are real); $\chi_1(k, u)$, $\chi_2(k, u)$ are functions which, for every fixed k, contain terms of higher order relative to u.

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Note: Figure translations are in progress. See original paper for figures.