Yu. L. Rodin

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Abstract

Full Text

Yu. L. Rodin

ON THE ALGEBRAIC THEORY OF ELLIPTIC SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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On a closed Riemann surface R of genus ρ we consider the equation

$$U_{\bar{z}} = CU + \overline{B}U, \tag{1}$$

where $C, B \in L_p,\ p > 2$. Solutions regular everywhere on R will be called **generalized constants**. Obviously, their number does not exceed two. As examples ([7]) show, it may also be smaller.

Theorem 1. *For the equation*

$$U_{\bar{z}} = BU - B\overline{U} \tag{2}$$

there exist exactly two generalized constants.

It is enough to show that the equation adjoint to (2),

$$V_{\bar{z}} = -BV + \overline{BV} \tag{3}$$

has 2ρ first-kind covariants ([7]). The latter follows from the fact that the solutions of the integral equation

$$V(z) + \frac{1}{\pi} \iint_R \left[\overline{B(t)}\,\overline{V(t)} - B(t)V(t)\right] A(z,t)\, dT = 0 \tag{4}$$

are regular on R, since the residue, in the polar with respect to z, of the kernel $A(z,t)$ (see ([5])) is purely imaginary, while the differential $\mathrm{Im}[V(z)\, dz]$ is closed on R.

Hence follows

Theorem 2. *Equation (1) can have only an even number (0 or 2) of generalized constants.*

In those cases where there are no generalized constants, we shall consider functions, regular everywhere on R, having multiplicative periods upon going around the contours $K_{2\mu}$ $(\mu = 1, 2, \ldots, \rho)$, some branch of which satisfies equation (1). Such functions we call **multiplicative generalized constants**. We normalize them by requiring that the branch under consideration take the value one at the point P_0.

In what follows we consider the surface $\widehat{R}$, obtained from R by making cuts along the cycles $K_{2\mu-1}$ $(\mu = 1, 2, \ldots, \rho)$ of a canonical basis.

Theorem 3. *For equation (1) there exists a unique normalized multiplicative generalized constant.*

Indeed, the indicated function satisfies the integral equation

$$U(z) = \exp\left\{-\frac{1}{\pi}\iint_{\widehat{R}}\left[C(t) + B(t)\overline{U(t)}/U(t)\right]A^*(t,z)\,dT\right\}, \tag{5}$$

where $A^*(t,z)$ is the Cauchy kernel, multivalued in z ([8]), normalized by the condition $A^*(t,P_0) \equiv 0$ on $\widehat{R}$. The operator on the right-hand side maps the sphere of sufficiently

of large radius in L_p, $p > 2$, into its compact subset, whence it follows that operator (5) has a fixed point and rotation $+1$ on the boundary of sphere (4). To establish the uniqueness of the generalized constant, we shall show that the index of any fixed point of this operator is equal to $+1$.

Taking the logarithm of (5) and introducing a new unknown function $\varphi(z) = -2\arg U(z)$, we arrive at the equation

$$\varphi(z) = \iint_{\widehat{R}}\{b(t,z) + M(t,z)\sin[\varphi(t) + m(t,z)]\}\,dT, \tag{6}$$

where

$$b(t,z) = \operatorname{Im}\frac{2}{\pi}C(t)A^*(t,z), \qquad m(t,z) = \arg\frac{2}{\pi}B(t)A^*(t,z),$$

$$M(t,z) = \frac{2}{\pi}|B(t)A^*(t,z)|.$$

The Fréchet derivative of operator (6) in a neighborhood of the fixed point $\varphi_0(z) = -2\arg U_0(z)$ has the form

$$K\psi(z) = \iint_{\widehat{R}} M(t,z)\cos[m(t,z) + \varphi_0(t)]\psi(t)\,dT. \tag{7}$$

We shall show that this operator has no eigenvalues on the segment $[0,1]$. We arrive at the integral equation

$$W(z) + \frac{\lambda}{\pi} \iint_{\hat{R}} B(t) \frac{\overline{U_0(t)}}{U_0(t)} [W(t) + \overline{W(t)}] A^*(t, z)\, dT = 0, \tag{8}$$

where $\operatorname{Re} W(z) = \psi(z)$.

By the Liouville theorem this equation has no single-valued solutions. Multivalued solutions must satisfy on the cuts $K_{2\mu-1}$ the boundary condition

$$W^+ - W^- = c_\mu, \qquad W_z = B\frac{\overline{U_0}}{U_0}[W + \overline{W}], \tag{9}$$

where c_μ are the periods of the solution along $K_{2\mu}$. In view of the fact that the imaginary part of any differential of the first kind satisfying the equation conjugate to (9) is cohomologous to a certain harmonic differential, one can show that $c_\mu = 0$ $(\mu = 1, 2, \ldots, \rho)$, which completes the proof.

In the case when the generalized constants are single-valued, the solutions of equation (1) turn out to be pseudoanalytic functions in the sense of L. Bers with generating pair $(1, U_1/U_0)$. Hence follows

Theorem 4. *The set of all generalized analytic functions (including multiplicatively multivalued ones) satisfying equation (1) is the product of the generalized constant $U_0(z)$ (in general multivalued) by the class of pseudoanalytic functions with generating pair $(1, U_1/U_0)$, where U_0 is the second generalized constant.*

Let us study the conditions for the existence of a generalized analytic function whose zeros and poles are determined by the divisor

$$D = \sum_i \alpha_i P_i - \sum_j \beta_j Q_j.$$

Put

$$U(z) = \Pi(z) W(z),$$

where

$$\Pi(z) = \exp\left\{ \sum_i \alpha_i \omega_{P_i P^0}(z) - \sum_j \beta_j \omega_{Q_j P^0}(z) \right\}, \tag{10}$$

P_0 is an arbitrary fixed point of the surface R, and $\omega(z)$ are integrals of the third kind ([2]).

Obviously, the problem formulated above is equivalent to the question of the existence of a multiplicative constant with periods inverse to the periods $\Pi(z)$, for the equation

$$W_z = CW + B_1\overline{W}, \qquad B_1 = B(z)\overline{\Pi}(z)/\Pi(z). \tag{11}$$

For the general elliptic system

$$w_z = q\overline{w}_{\bar{z}} + C_0 W + B_0 \overline{W} \tag{12}$$

the same problem is reduced to the case of equation (1) by means of the substitution

$$U = w - \overline{q}\, w, \tag{13}$$

where

$$C = \frac{C_0 + B_0 q - \overline{q}\, q_{\bar{z}}}{1 - |q|^2}, \qquad B = \frac{C_0 q + B_0 - q_{\bar{z}}}{1 - |q|^2}. \tag{14}$$

Hence there follows

Theorem 5 (Abel). *In order that there exist a solution of equation (12) whose zeros and poles are determined by the divisor D, it is necessary and sufficient that the following system be solvable in integers*

$$\sum_{\nu=1}^{\rho} (A_{\mu\nu} h_\nu - B_{\mu\nu} l_\nu) = \sum_i \alpha_{iZ_\mu}(P_i) - \sum_j \beta_{jZ_\mu}(Q_j)$$

$$-\frac{1}{\pi} \iint_R \left[C(t) + B(t) \frac{\overline{\Pi}(t)\overline{W}_0(t)}{\Pi(t) W_0(t)} \right] Z'_\mu(t)\, dT \qquad (\mu = 1, \dots, \rho), \tag{15}$$

where dZ_μ $(\mu = 1, \dots, \rho)$ is a basis of analytic differentials of the first kind on the surface R, and $A_{\mu\nu}, B_{\mu\nu}$ are the periods of the differentials dZ_μ along the cycles of the canonical basis, and $W_0(z)$ is a multiplicative constant for equation (11).

The form $\varphi = \alpha\, dz + \beta\, d\overline{z}$ is called a **differential adjoint to equation (12)** if

$$\beta = \overline{q\alpha}, \qquad \alpha_{\overline{z}} = \beta_z - C_0 \alpha - \overline{B}_0 \overline{\alpha}. \tag{16}$$

This definition obviously includes the cases of quasi-analytic (9) and generalized analytic (7) differentials. As a simple calculation shows, the differential $\alpha\, dz$ in this case turns out to be a generalized analytic differential belonging to the equation $\alpha_{\overline{z}} = -C\alpha - \overline{B}\, \overline{\alpha}$, adjoint to (1). From work (7) it follows

Theorem 6 (Riemann–Roch). *The difference between the number A of solutions of equation (12) that are multiples of the divisor $-\Delta$, and the number B of differentials adjoint to (12) that are multiples of Δ, is equal to*

$$A - B = 2\,\mathrm{ord}(\Delta) - 2\rho + 2. \tag{17}$$

The differentials introduced can, in an obvious way, be used for solving boundary-value problems.

As examples (7) show, the number $2\rho_1$ of differentials of the first kind adjoint to equation (12) may be equal to 2ρ or to $2\rho - 2$.

Theorem 7 (Noether). *Suppose the question is studied of the existence of a solution of equation (12) having poles at most at the points P_1, P_1P_2, $P_1P_2P_3$, etc., and at the indicated point a pole is obligatory. Then, to the question of the existence of each sought solution separately, we shall obtain negative answers exactly ρ_1 times (cf. (3)).*

In particular, from this follows Weierstrass' s gap theorem (3) for system (12). In the usual way, the Weierstrass points for system (12) are introduced.

Theorem 8. The number of Weierstrass points for the system (12) is finite and, for $\rho > 1$, is different from zero.

We shall call the **weight of a point** P the number

$$\tau_P = \sigma_1 + \sigma_2 + \cdots + \sigma_{\rho_1} - \frac{\rho_1(\rho_1 + 1)}{2}$$

(σ_k are the numbers of the "gaps" at this point). Obviously, P is a Weierstrass point if $\tau_P > 0$.

The relation

$$\sum_{P \in R} \tau_P = (\rho - 1)\rho(\rho + 1), \tag{18}$$

is proved, whence the assertion of the theorem follows.

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Note: Figure translations are in progress. See original paper for figures.