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Abstract

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ON A CERTAIN CLASS OF SPECIAL AUTOMORPHISMS AND SPECIAL FLOWS

(Presented by Academician A. N. Kolmogorov on 20 VI 1963)

The notion of a K-system was first introduced by A. N. Kolmogorov in [1]. In the present note we indicate some conditions under which special automorphisms and special flows (see below) are K-systems. Verification of these conditions in one concrete case (§ 3) makes it possible to obtain a number of examples of K-flows with finite entropy.

§ 1. **Special automorphisms.** Let T be an automorphism of a Lebesgue space X with measure μ, and let $f(x)$, $x \in X$, be a measurable positive function taking integer values $a_1, a_2, \ldots$, with

$$\int_X f(x)\, d\mu < \infty.$$

Denote by U the set of all integers, and assign to each point $u \in U$ a measure equal to 1. In the direct product of the spaces with measure $X \times U$, select the subspace $\widetilde{X}$ consisting of those points (x, u) for which $0 \leq u \leq f(x) - 1$. Dividing the measure in $\widetilde{X}$ by the number

$$\int_X f(x)\, d\mu,$$

we obtain a normalized measure. Denote it by $\tilde{\mu}$. Define a transformation $\widetilde{T}$ of the space $\widetilde{X}$ by the formula

$$\widetilde{T}(x,u) = \begin{cases} (x, u+1), & \text{if } u < f(x) - 1, \\ (Tx, 0), & \text{if } u = f(x) - 1. \end{cases}$$

It is not hard to verify that $\widetilde{T}$ is an automorphism of the space $\widetilde{X}$ with measure $\tilde{\mu}$. It is called a **special automorphism, built over the automorphism** T **and the function** $f(x)$.

Assume now that T is a K-automorphism [1], i.e., there exists a measurable partition ξ_0 of the space X with the following properties:

$$1)\quad T^n\xi_0 = \xi_n \geq \xi_{n-1}; \qquad 2)\quad \prod_{n=-\infty}^{\infty} T^n\xi_0 = \varepsilon; \qquad 3)\quad \bigcap_{n=-\infty}^{\infty} T^n\xi_0 = \nu,$$

where ε is the partition of the space X into individual points, and ν is the trivial partition, whose only element is all of X. A partition with properties 1)-3) is called a **K-partition corresponding to the K-automorphism** T. Throughout the whole work we shall assume that the function $f(x)$ is constant mod 0 on the elements of the partition ξ_0.

Let γ_n be the partition of the space X into the preimages of the values of the function

$$S_n(x) = f(x) + f(Tx) + \dots + f(T^n x), \qquad n \geq 1.$$

Put $\eta_n = \gamma_{n-1} \cdot \xi_{-n}$. It is easy to verify that $\eta_n \leq \eta_{n-1}$. We shall call the natural numbers $a_1, a_2, \dots$ **relatively prime** if their greatest common divisor is equal to 1.

Theorem 1. *If the function $f(x)$ takes relatively prime values and the condition*

$$\bigcap_{n=1}^{\infty} \eta_n = \nu, \tag{*}$$

is fulfilled, then $\widetilde{T}$ is a K-automorphism.

Proof. Let $\tilde{\xi}_0$ be the partition of the space $\widetilde{X}$ obtained from the partition ξ_0 of the space X by the following rule: the points $(x_1, u_1) \in \widetilde{X}$ and $(x_2, u_2) \in \widetilde{X}$ belong to the same element of the partition $\tilde{\xi}_0$ if the points $x_1 \in X$ and $x_2 \in X$ belong to the same element of the partition ξ_0 and $u_1 = u_2$. The partition $\tilde{\xi}_0$ is, obviously, measurable. We shall show that, if the hypotheses of the theorem are satisfied, then $\tilde{\xi}_0$ is a K-partition corresponding to the automorphism $\widetilde{T}$. Properties 1) and 2) follow at once from the corresponding properties of the partition ξ_0 and the constancy of the function $f(x)$ on the elements of this partition. Only property 3) requires verification.

Denote by β the partition of the space $\widetilde{X}$ into sets B_i of the form

$$B_i = \{(x, u) : x \in X,\ u = i\}.$$

Since $\tilde{\mu}(B_0) > 0$, for each n one can consider the partition ξ'_n of the set B_0 formed by the intersections of the elements of the partition $\tilde{\xi}_n$ with B_0. Define another sequence $\{\eta'_n\}$, $n > 0$, of partitions of the set B_0, assigning the points $(x_1, 0) \in B_0$ and $(x_2, 0) \in B_0$ to the same element of the partition η'_n if and only if x_1 and x_2 belong to the same element of the partition η_n. All the constructed partitions are measurable and, for $n > 0$, satisfy the inequalities

$$\xi'_{-n} \leq \xi'_{-n+1}, \qquad \eta'_n \leq \eta'_{n-1}.$$

It is also clear that, by virtue of (*),

$$\bigcap_{n>0} \eta'_n = \nu^*.$$

We shall prove that

$$\bigcap_{n<0} \xi'_n = \nu. \tag{1}$$

If $\sup_{x\in X} f(x) = M < \infty$, then (1) follows from the easily verified inequality

$$\xi'_{-nM} \leq \eta'_n, \qquad n > 0.$$

In the case of an unbounded function $f(x)$, one can choose a sequence of integers $k_n \to \infty$ and a sequence of sets $A_n \subset B_0$ so that

$$\tilde{\mu}(A_n) \xrightarrow[n\to\infty]{} \tilde{\mu}(B_0)$$

and on the set A_n the inequality $\xi'_{-k_n} \leq \eta'_n$ holds; whence (1) again follows.

To complete the proof of the theorem, we derive from (1), using the mutual independence of the values of the function $f(x)$, property 3) of the partition $\tilde{\xi}_0$. First note that the partition β is finite or countable. Then

$$\bigcap_n \beta \cdot \tilde{\xi}_n = \beta \cdot \bigcap_n \tilde{\xi}_n.$$

Taking this into account, it is not difficult to obtain from (1) the inequality

$$\bigcap_n \tilde{\xi}_n \leq \beta.$$

Let B be the element of the partition $\bigcap_n \tilde{\xi}_n$ containing B_0. The partition $\bigcap_n \tilde{\xi}_n$ is invariant with respect to the automorphism $\widetilde{T}$. Hence, for any k, the set $\widetilde{T}^{kB}$ is an element of the partition $\bigcap_n \tilde{\xi}_n$. Since the automorphism $\widetilde{T}$ is ergodic** and $\tilde{\mu}(B) > 0$, we have

$$d = \min_{k>0}(k : \widetilde{T}^{kB} = B) < \infty.$$

Then $\widetilde{T}^{kB} = B \bmod 0$ for $k = nd$, and

$$\widetilde{T}^{kB} \cap B = \emptyset \bmod 0$$

for $k \neq nd,\ n = 0, \pm1, \pm2, \ldots.$ But

* The set B_0 may be regarded as a subspace of the space X; ν denotes here the partition of B_0 whose only el

** A special automorphism constructed from an ergodic automorphism and any function is, as is easy to see, e

$\mu(\widetilde{T}^{a_i}B_0 \cap B_0) > 0$ for $i = 1, 2, \ldots$; hence, $\widetilde{T}^{a_i}B = B \bmod 0$, $i = 1, 2, \ldots$. It is clear from this that $d = 1$, since the numbers a_i are relatively prime. Consequently, the set B is invariant with respect to $\widetilde{T}$, and therefore $\hat{\mu}(B) = 1$, and B is the only $\bmod 0$ element of the partition $\bigcap \xi_n$.

Let us give an example in which the function $f(x)$ takes the values 1 and 2, but condition $(*)$ is not satisfied and $\widetilde{T}$ is not a K-automorphism. Let T be a Bernoulli automorphism ([3]) with state space consisting of two points, 0 and 1, of equal measure, and

$$f(x) = \begin{cases} 1, & \text{if } x_{-1} = 1,\ x_0 = 0 \text{ or } x_0 = 1,\ x_1 = 0; \\ 2, & \text{if } x_{-1} = 0,\ x_0 = 0 \text{ or } x_0 = 1,\ x_1 = 1. \end{cases}$$

As ξ_0 one may take the partition generated by the values of x_i for all $i \leqslant 1$. Setting $A = \{x : x_0 = 1,\ x_1 = 0\}$, it is not difficult to verify that $\lim_{n\to\infty} \mu(A/\eta_n) = 0$ or 1. Hence condition $(*)$ is not satisfied for the function $f(x)$. The automorphism $\widetilde{T}$ is easily realized as a shift in the space of trajectories of a certain Markov chain with six states. This chain has two subclasses, whence it is clear that $\widetilde{T}$ is not a K-automorphism.

At the same time, the automorphism $\widetilde{T}_1$, constructed from T and the function $f_1(x) = 3 - f(x)$, is a K-automorphism, since it is isomorphic to a shift in the space of realizations of a Markov chain with six states having only one subclass. For each $n > 0$, the partition γ_n^1 of the space X into preimages of the values of the function $S_n^1(x) = f_1(x) + f_1(Tx) + \cdots + f_1(T^n x)$ coincides with γ_n, and consequently condition $(*)$ is also not satisfied for the function $f_1(x)$. This example shows that condition $(*)$ is not necessary in order that $\widetilde{T}$ be a K-automorphism.

§ 2. **Special flows.** Keeping the assumptions and notation of § 1 concerning the space X and its automorphism T, consider the special flow $\{S_t\}$ ([3]), constructed from T and an integrable function $f(x)$, $x \in X$, bounded below by a positive constant. The space in which the flow $\{S_t\}$ acts consists of those points (x, u) of the direct product $X \times (u)$ of the space X with the numerical line (u), with the usual Lebesgue measure, for which $0 \leqslant u < f(x)$. By definition, for $t < \inf_{x\in X} f(x)$,

$$S_t(x, u) = \begin{cases} (x, u + t), & \text{if } t < f(x) - u, \\ Tx,\ t + u - f(x), & \text{if } t \geqslant f(x) - u. \end{cases}$$

For the remaining t, the automorphism S_t can be defined from the fact that $\{S_t\}$ is a group.

We require, as in § 1, that the function $f(x)$ take a finite or countable number of values and be constant $\bmod 0$ on the elements of the partition ξ_0. The notation η_n will then have the same meaning as in § 1.

Positive numbers $\alpha_1, \alpha_2, \ldots$ will be called incommensurable if the totality of all possible finite sums of the form $\sum_i n_i \alpha_i$, with integer coefficients n_i, is everywhere dense on the numerical line. For incommensurability it is sufficient, for example, that among the α_i there are two numbers whose ratio is irrational, or that the set $\{\alpha_i\}$ contain an infinite bounded subset.

Theorem 2. *If the values of the function $f(x)$ are incommensurable and condition $(*)$ is fulfilled (see § 1), then $\{S_t\}$ is a K-flow.*

The proof of this theorem is similar to the proof of Theorem 1.

§ 3. Examples. In this section, by T we shall mean the shift in the space X of trajectories of a homogeneous ergodic Markov chain $Z = \{\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots\}$ with a finite number of states $\omega_1, \omega_2, \ldots, \omega_s$. Assuming the initial distribution to be stationary, it is easy to show that T is a K-automorphism, while the partition ξ_0, generated by the random variables $x_i,\ i \le 0$, is a K-partition. Denote by F_0 the collection of positive functions $f(x)$, $x \in X$, measurable with respect to the random variable x_0.

Theorem 3. *In order that the special automorphism constructed from T and any function $f \in F_0$ with mutually coprime values be a K-automorphism, it is necessary and sufficient that the Markov chain Z satisfy the local limit theorem in the Kolmogorov form* ([4]).

The proof of this theorem is not difficult and rests mainly on the following known facts:

1) The special automorphism constructed from T and $f \in F_0$ is the shift in the space of realizations of some stationary Markov chain with a finite number of states.

2) Ergodicity of a Markov chain with a finite number of states consists in the existence of a single essential class of states containing no subclasses.

3) An ergodic Markov chain with a finite number of states satisfies the local limit theorem ([4]) if and only if its rank (([5]), $p.87$) is equal to the number of states.

Theorem 4. *In order that the automorphism T and any function $f \in F_0$ satisfy condition $(*)$, it is necessary and sufficient that the Markov chain Z satisfy the local limit theorem in the Kolmogorov form* ([4]).

The necessity follows immediately from Theorems 1 and 3. In proving sufficiency, a number of asymptotic formulas derived in ([4]) are used.

Comparison of Theorems 2 and 4 leads to the following result.

Theorem 5. *If the Markov chain Z satisfies the local limit theorem in the Kolmogorov form* ([4]) *and the function $f(x) \in F_0$ takes incommensurable values, then the special flow constructed from T and f is a K-flow.*

In conclusion, I take this opportunity to express my gratitude to Ya. G. Sinai, under whose supervision this work was carried out.

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Note: Figure translations are in progress. See original paper for figures.