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# MATHEMATICAL PHYSICS

1963

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## Abstract

## Full Text

# MATHEMATICAL PHYSICS

## V. A. BOROVIKOV

# ON THE THREE-DIMENSIONAL PROBLEM OF DIFFRACTION BY A PRISM\*

(Presented by Academician V. I. Smirnov, 4 VIII 1962)

1. In this note the author's results <sup>(1)</sup> are generalized to the case of the three-dimensional problem of diffraction by an infinite prism  $S$ , whose base is a convex polygon. We shall construct the Green function  $\Gamma(a, x, t)$ , where  $a$  and  $x$  are points of three-dimensional space, for the wave equation and zero boundary conditions on  $S$ , and shall indicate the form of the singularities of  $\Gamma(a, x, t)$  as a function of  $t$  for fixed  $a, x$ . The Fourier transform with respect to  $t$  of the function  $\Gamma(a, x, t)$  is the Green function for the Helmholtz equation and the boundary-value problem  $u|_S = 0$ ; its asymptotics as  $k \rightarrow \infty$  is determined by the singularities of the function  $\Gamma(a, x, t)$ .
2. As in <sup>(1)</sup>, of fundamental importance is the construction of the Green function for the case when  $S$  is a dihedral angle. We shall indicate what form the Green function has in this case.

Introduce cylindrical coordinates  $z, r, \varphi$ , so that the region in which we solve the wave equation is determined by the inequality  $0 \leq \varphi \leq \Phi$  ( $\pi > \Phi \geq 2\pi$ ), and the boundary conditions have the form  $u(z, r, 0) = u(z, r, \Phi) = 0$ . Let  $z = 0, r = R, \varphi = \varphi_0$  ( $0 < \varphi_0 < \Phi$ ) be the coordinates of the source  $a$ . Then the region in which the wave scattered by the edge  $S$  will appear has the form

$$\Omega: t > \sqrt{(R+r)^2 + z^2}. \quad (1)$$

Outside this region the solution is represented as the sum of the incident wave

$$\frac{\delta\left(t - \sqrt{R^2 + r^2 - 2Rr \cos(\varphi - \varphi_0) + z^2}\right)}{4\pi t} \quad (2)$$

and reflected waves, which are constructed according to the usual laws of geometrical optics.

The solution inside the region  $\Omega$  was given by G. I. Petrashen' and his students in <sup>(2)</sup>. Namely, inside  $\Omega$  the solution is represented in the form

$$u = u_{\varphi - \varphi_0 - \pi} + u_{\varphi + \varphi_0 + \pi} - u_{\varphi - \varphi_0 + \pi} - u_{\varphi + \varphi_0 - \pi}, \quad (3)$$

where for the function  $u_\beta$  there is a representation in the form of a double integral (see (2), formulas (70.5)); in order to obtain the solution of our problem, these formulas must be differentiated with respect to  $t$ . It is not difficult, however, to show by direct verification that for  $u_\beta$  the formula holds

$$u_\beta = \frac{1}{8\pi\Phi} \frac{1}{2Rr \operatorname{sh} \gamma} \frac{\sin \frac{\pi}{\Phi} \beta}{\sin^2 \frac{\pi}{2\Phi} \beta + \operatorname{sh}^2 \frac{\pi}{2\Phi} \gamma}, \quad (4)$$

where

$$\operatorname{ch} \gamma = \frac{t^2 - R^2 - r^2 - z^2}{2Rr}; \quad \gamma > 0.$$

3. Knowing the Green function for the case when  $S$  is a dihedral angle, it is easy to construct the Green function for an arbitrary prism  $S$ , but only for small times  $t$ , and then to pass to arbitrary  $t$  by—

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\* Reported at the Symposium on Wave Diffraction in the city of Gorky in June 1962.

by a sequence of convolutions of Green' s functions (see (1), Sec. 2). These considerations make it possible to find the singularities of the Green' s function. For definiteness, let us take the case when the projection of the source  $a$  onto the plane perpendicular to the edges of the prism is situated as shown in Fig. 1. Consider the process of propagation of a wave from the point  $a$ . First, at the point  $a$  a spherical wave is excited, described for small  $t$  by formula (2) (if one chooses a cylindrical coordinate system, as was done in Sec. 2, attached to the edge  $A$ ). As  $t$  increases, the front of the incident wave reaches the edge  $A$ , and the incident wave, scattering at  $A$ , produces a wave which may be called the diffracted wave of the first order. The equation of the front of this wave has the form

(Figure: Fig. 1)

**Fig. 1**

$$t = \sqrt{(R + r)^2 + z^2}$$

and it is expressed by formulas (3), (4).

When the front of the diffracted wave of the first order reaches the edge  $B$ , this wave, scattering at the edge  $B$ , will give a diffracted wave of the second order, whose front is expressed by the equation

$$t = \sqrt{(R + h + r)^2 + z^2},$$

where  $h$  is the width of the face  $AB$ , and  $r$  is the distance to the edge  $B$ .

The diffracted wave of the second order, scattering in turn at the edges  $A$  and  $C$ , produces waves of the third order, and so on.

The Green's function  $\Gamma(a, x, t)$  is the sum of the incident wave, the reflected wave, and all the diffracted waves, and has singularities at those instants of time  $t$  at which the front of any of these waves passes through the point  $x$  under consideration. To determine the singularities of the Green's function, one must know the behavior of the diffracted waves in a neighborhood of their fronts. The following problem arises: from the behavior of a diffracted wave in a neighborhood of its front, determine the behavior of the wave generated by it.

4. Let us indicate the general form of a diffracted wave. The front of any diffracted wave can be written in the form

$$t = \sqrt{(t_0 + r)^2 + z^2}, \quad (5)$$

where  $t_0$  is the instant of formation of the wave, and  $r$  is the distance from the point under consideration to the edge at which the wave arose. For investigating the diffracted wave in a neighborhood of the front, it is expedient to use the ray method (see (3)). Denote by  $z, r, \varphi$  the local coordinate system attached to the edge of the prism under consideration (for example, to the edge  $A$ ). Let the edge go along the axis  $r = 0$ , and let the face  $AB$  of the prism have the equation  $\varphi = 0$ ,  $0 < r < h$ . Introduce ray coordinates corresponding to the wave fronts (5):

$$\tau = \sqrt{(t_0 + r)^2 + z^2}; \quad \theta = \arctg \frac{z}{t_0 + r}; \quad \varphi.$$

Then any wave whose front has the form (5) can be written in the form

$$\frac{(t - \tau)^\lambda}{\Gamma(\lambda + 1)} \frac{w_0(\varphi, \theta)}{\sqrt{\tau(\tau \cos \theta - t_0)}} + \frac{(t - \tau)^{\lambda+1}}{\Gamma(\lambda + 2)} \left[ -\frac{\partial^2 w_0}{\partial \theta^2} \frac{1}{2t^{3/2}(\tau \cos \theta - t_0)^{1/2}} - \frac{w_0}{8\tau^{3/2}(\tau \cos \theta - t_0)^{1/2}} - \frac{w_0}{8 \cos \theta \tau^{1/2}(\tau \cos \theta - t_0)} \right] \quad (6)$$

Here  $w_0(\varphi, \theta)$  and  $w_1(\varphi, \theta)$  are, generally speaking, arbitrary functions. It is easy to show that for any diffraction wave

$$\left. \frac{\partial^{2k} w_0(\varphi, \theta)}{\partial \varphi^{2k}} \right|_{\varphi=0} = \left. \frac{\partial^{2k} w_1(\varphi, \theta)}{\partial \varphi^{2k}} \right|_{\varphi=0} = 0 \quad (k = 0, 1, 2, \dots).$$

5. The wave arising at the edge  $A$ , described by formulas (6), produces, upon scattering at the edge  $B$ , a diffraction wave of the next order, described by certain parameters  $\lambda^{(1)}$ ,  $w_0^{(1)}(\varphi, \theta)$ ,  $w_1^{(1)}(\varphi, \theta)$  (where the coordinates

$r, \varphi, \theta$  are tied to the edge  $B$ , on the face  $BC$   $\varphi = 0$ , and  $t_0$  should be replaced by  $t_0 + h$ , where  $h$  is the width of the face  $AB$ ). We shall indicate the relation between  $\lambda, w_0, w_1$  and  $\lambda^{(1)}, w_0^{(1)}, w_1^{(1)}$ . Then, by the recurrent formulas given, we shall be able to determine the parameters of a diffraction wave of any order. Let the magnitude of the dihedral angle  $ABC$  be  $2\pi - \Phi_B$ , where  $\pi < \Phi_B < 2\pi$ , and

$$w_0(\varphi, \theta) = \varphi a(\theta) + \frac{\varphi^3}{6} b(\theta) + O(\varphi^5); \quad w_1(\varphi, \theta) = \varphi e(\theta) + O(\varphi^3).$$

$$\lambda^{(1)} = \lambda - \frac{3}{2}.$$

Then

$$w_0^{(1)}(\varphi, \theta) = \frac{\pi a(\theta) u(\varphi, \Phi_B)}{4\sqrt{2} h^{1/2} (\cos \theta)^{3/2}},$$

$$w_1^{(1)}(\varphi, \theta) = -\frac{\pi a(\theta) u''_{\varphi\varphi}(\varphi\Phi_B)}{8\sqrt{2} h^{5/2} (\cos \theta)^{5/2}} - \frac{5\pi a(\theta) u(\varphi, \Phi_B)}{32\sqrt{2} (\cos \theta)^{5/2} h^{5/2}}$$

$$+ \frac{\pi e(\theta) u(\varphi, \Phi_B)}{4\sqrt{2} h^{3/2} (\cos \theta)^{3/2}} - \frac{\pi b(\theta) u(\varphi, \Phi_B)}{8\sqrt{2} h^{5/2} (\cos \theta)^{5/2}}.$$

Here

$$u(\varphi, \Phi) = \frac{\sqrt{\pi}}{\Phi^2} \left[ \frac{1}{\sin^2 \frac{\pi}{2\Phi}(\varphi - \pi)} - \frac{1}{\sin^2 \frac{\pi}{2\Phi}(\varphi + \pi)} \right]. \quad (7)$$

6. The Fourier transform with respect to  $t$  of the function  $\Gamma(a, x, t)$

$$\Gamma_1(a, x, k) = \int_{-\infty}^{\infty} e^{ikt} \Gamma(a, x, t) dt \quad (8)$$

is the Green's function for the stationary problem of diffraction by a prism. To justify this assertion it is enough to show that the function  $\Gamma(a, x, t)$  and its derivatives decrease sufficiently rapidly as  $t \rightarrow +\infty$  (for  $t < 0$ ,  $\Gamma(a, x, t) = 0$ ). We do not possess a proof of the decrease of  $\Gamma(a, x, t)$ ; however, on physical grounds it should not raise doubts.

If one assumes that integral (8) can be integrated by parts the required number of times, then from the results formulated above concerning the singularities of the function  $\Gamma(a, x, t)$  (more precisely, the singularities of the diffraction waves composing it) it is easy to obtain the first terms of the asymptotics of the function  $\Gamma_1(a, x, k)$  as  $k \rightarrow \infty^*$ .

It turns out that to each optical path connecting the points  $a$  and  $x$  there corresponds its own contribution to the asymptotics of  $\Gamma_1(a, x, k)$ , beginning with terms of order  $k^{-3/2n+1}$ , where  $n$  is the number of refractions of the optical path.

By an optical path we shall mean any broken line connecting the points  $a$  and  $x$  and satisfying the following conditions: a) the first (last) segment of the optical path connects the point  $a$  (the point  $x$ ) with one of the edges  $S$ ,

\* If one assumes that  $\text{Im } k > 0$ , then integral (8) certainly converges, and all conclusions of §§ 6,7 are rigorous

having no more common points with  $S$ ; b) each inner segment of the optical path lies on one of the faces of the prism, joining its opposite edges; c) the segments of the optical path form with the edges of the prism equal angles, depending only on the trace of the optical path.

7. Let us consider the simplest example. Let the projection of the source  $a$  and of the observation point  $x$  onto a plane perpendicular to the edges of the prism have the form shown in Fig. 2. Let  $\theta$  be the angle formed by the optical path under consideration from the point  $a$  to the point  $x$  with the edges of the prism, and let  $T$  be the length of this path. The meaning of the remaining parameters is clear from Fig. 2. Let the function  $u(\varphi, \Phi)$  be defined by formula (7). Then the first term of the asymptotics of the contribution to  $\Gamma_1(a, x, k)$ , corresponding to the optical path  $aABx$ , has the form

(Figure: Fig. 2)

**Fig. 2**

$$-\frac{e^{ikT}}{k^2} \frac{\pi u(\varphi_B, \Phi_B) u(\varphi_a, \Phi_A)}{64 L_{AB}^{3/2} (\cos \theta)^{3/2} R_A^{1/2} R_B'^{1/2} T^{1/2}}. \quad (9)$$

The first term of the asymptotics of the contribution corresponding to the optical path  $aABCx$  has the form:

$$i \frac{e^{ikT}}{k^{7/2}} \frac{\pi^2 u(\varphi_a, \Phi_A) u'_\varphi(0, \Phi_B) u(\varphi_C, \Phi_C)}{256 \sqrt{2} L_{AB}^{3/2} L_{BC}^{3/2} (\cos \theta)^3 R_a'^{1/2} R_c'^{1/2} T^{1/2}}. \quad (10)$$

Expressions have also been obtained for the second terms of the asymptotics; we do not give them because of their bulkiness.

For the arrangement of the points  $a$  and  $x$  shown in Fig. 2, the first term of the asymptotics of  $\Gamma_1(a, x, k)$  is determined by formula (10) and is of order  $k^{-2}$ . If, however, the point  $x$  passes from region  $I$  into region  $II$  (see Fig. 2), then

only the optical path  $aABCx$  (and paths with a larger number of refractions) remains, so that the first term of the asymptotics of  $\Gamma_1(a, x, k)$  begins with terms of order  $k^{-7/2}$  and is determined by formula (10). In the penumbra region (the boundary between regions  $I$  and  $II$ ) the solution can be represented in a form analogous to Fresnel integrals.

All the considerations of the note can be carried over to the case of the boundary condition  $\partial u / \partial h = 0$ .

Received  
20 VI 1962

## CITED LITERATURE

1. V. A. Borovikov, *DAN*, **144**, No. 4, 743 (1962).
2. G. I. Petrashen' , B. G. Nikolaev, D. P. Kouzov, *Uch. zap. LGU*, No. 246, 5 (1958).
3. A. S. Alekseev, V. M. Babich, B. Ya. Gel' chinskii, "Problems of the dynamic theory of propagation of seismic waves," Collection V, Leningrad State University Press, 1961, p. 3.

*Note: Figure translations are in progress. See original paper for figures.*

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