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Abstract

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ASYMPTOTICS OF CERTAIN ROTATIONAL MOTIONS

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§ 1. Statement of the problem.

In papers (^{1}, ^{2}), systems with one degree of freedom were considered for which the solution of the unperturbed system describes an oscillatory process. For them, amplitude curves, the period, and a number of other parameters were found in the first and second approximations. It is of interest to analyze the rotational regimes of such systems. General methods for calculating oscillatory and rotational motions, which are used in the present paper, were developed in (^{3}–^{5}). In papers (^{6}, ^{7}), similar problems were considered by another method.

Let the unperturbed system have the form

$$m(x)\ddot{y} + Q(x,y) = 0, \qquad x = \text{const}\,. \tag{1}$$

Here y is a one-dimensional coordinate; $m(x)$ is the mass; $x = (x_1, \dots, x_n)$ is a set of parameters; $Q(x,y) \equiv \partial V(x,y)/\partial y$ is a potential force causing rotation; $Q(x,y)$ is a periodic function of y with period 2π, and

$$\int_0^{2\pi} Q(x,y)\,dy = 0.$$

The perturbed system corresponding to (1) is written in the form:

$$\frac{d}{dt}\,[m(x)\dot{y}] + Q(x,y) = \varepsilon f(x,y,\dot{y}), \qquad \dot{x} = \varepsilon X(x,y,\dot{y}), \tag{2}$$

where ε is a small parameter, and f and X are periodic in y. We restrict ourselves here to the first approximation; therefore in (2) terms of order ε^2 and higher have been discarded. For the unperturbed system (1) the energy $E = m(x)\dot{y}^2/2 + V(x,y)$ is constant. We pose the problem: to find, in the first approximation, a slowly varying function $E(\varepsilon,t)$ describing in the first approximation the energy of the perturbed motion, and also to find, in the same approximation, the slowly varying parameters x.

§ 2. Main results.

In (^{3}-^{5}), systems close to Hamiltonian ones of the form

$$\dot{q} = \frac{\partial H(p,q,x)}{\partial p} - \varepsilon f^{(p)}(p,q,x,\varepsilon),$$

$$\dot{p} = -\frac{\partial H(p,q,x)}{\partial q} + \varepsilon f^{(q)}(p,q,x,\varepsilon), \tag{3}$$

$$\dot{x} = \varepsilon X(p,q,x,\varepsilon).$$

Here $p = (p_1, \dots, p_m)$; $q = (q_1, \dots, q_m)$; $x = (x_1, \dots, x_n)$; $H(p,q,x)$ is the Hamiltonian function of the corresponding unperturbed system obtained from (3) for $\varepsilon = 0$; $f^{(p)}(p,q,x,\varepsilon)$, $f^{(q)}(p,q,x,\varepsilon)$, $X(p,q,x,\varepsilon)$ are vector functions of the corresponding dimensions. In (^{3}-^{5}), equations were derived which describe, in the first approximation, the change of the energy E, the action I, and the parameters $x = (x_1, \dots, x_n)$:

$$\dot{E} = \frac{\varepsilon}{T_0} \int_{T_0} \left(\frac{\partial H}{\partial x} X_1 + f_0^{(p)} \dot{p} + f_0^{(q)} \dot{q} \right) dt; \tag{4}$$

$$\dot{I} = \varepsilon \int_{T_0} \left[f_1^{(p)} \dot{p} + f_0^{(q)} \dot{q} + \frac{\partial H}{\partial x} (X_1 - \overline{X}_1) \right] dt; \tag{5}$$

$$\dot{x} = \frac{\varepsilon}{T_0} \int_{T_0} X_1 \, dt \tag{6}$$

$$\left(X_1 \equiv X \big|_{\varepsilon=0}, \quad f_0^{(p)} \equiv f^{(p)} \big|_{\varepsilon=0}, \quad f_0^{(q)} \equiv f^{(q)} \big|_{\varepsilon=0} \right),$$

where the integrals are taken over a segment of the unperturbed phase trajectory in the space p, q, x, corresponding to the time increment $\Delta t = T_0$ (T_0 is the period of oscillations or rotations) and to the energy level E. Equations (2) are a special case of system (3). After transformations, from (4) we obtain the energy equation for the rotational motion (2) in the form

$$\dot{E} = \frac{\varepsilon}{T_0}\int_0^{2\pi} f\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right) dy +$$
$$+ \frac{\varepsilon}{T_0}\int_0^{2\pi}\left(-E\frac{\partial m}{\partial x} + \frac{\partial mV}{\partial x}\right) X\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right)\frac{dy}{\sqrt{2m(x)(E - V(x,y))}},$$
$$\dot{x} = \frac{\varepsilon}{T_0}\int_0^{2\pi}\frac{X\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right)}{\sqrt{\frac{2}{m(x)}(E - V(x,y))}}\, dy, \tag{7}$$

where

$$T_0 = \sqrt{\frac{m(x)}{2}}\int_0^{2\pi}\frac{dy}{\sqrt{E - V(x,y)}}$$

is the period of rotation. Expressing the action integral I in terms of the energy E, from (5) we obtain the second form of the equation describing the change of action or energy:

$$\dot{I} = \varepsilon\int_0^{2\pi} f\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right) dy + \varepsilon\int_0^{2\pi}\left(-E\frac{\partial m}{\partial x} + \frac{\partial mV}{\partial x}\right)\times$$
$$\times\left\{X\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right) - \frac{1}{T_0}\int_0^{2\pi}\frac{X\left(x, y, \sqrt{\frac{2}{m(x)}(E - V(x,y))}\right) dy}{\sqrt{\frac{2}{m(x)}(E - V(x,y))}}\right\}\frac{dy}{\sqrt{2m(x)(E - V(x,y}} \tag{8}$$

Here

$$I = \int_0^{2\pi}\sqrt{2m(x)(E - V(x,y))}\, dy$$

is the action integral. It is not difficult to verify the equivalence of these two forms.

§ 3. **Special cases.** Let the only slow parameter x be the "slow" time $\tau = \varepsilon t$. Then the energy and action equations (7), (8) can be written in the following equivalent forms:

$$\frac{dE}{d\tau} = \frac{1}{T_0}\int_0^{2\pi} f\left(\tau, y, \sqrt{\frac{2}{m(\tau)}(E - V(\tau,y))}\right) dy +$$

$$+\frac{1}{T_0}\int_0^{2\pi}\left(-E\frac{\partial m}{\partial\tau}+\frac{\partial mV}{\partial\tau}\right)\frac{dy}{\sqrt{2m(\tau)(E-V(\tau,y))}}, \tag{9}$$

$$\frac{dI}{d\tau}=\int_0^{2\pi} f\left(\tau, y, \sqrt{\frac{2}{m(\tau)}(E-V(\tau,y))}\right)dy.$$

In a number of cases equation (9) is integrated elementarily. Thus, if f and Q are odd in y and l-periodic in y with period 2π, then

$$I=\int_0^{2\pi}\sqrt{2m(\tau)(E-V(\tau,y))}\,dy=\text{const}.$$

If $f(\tau,y,\dot{y})=-2b(\tau)\dot{y}$, then we obtain:

$$\int_0^{2\pi}\sqrt{2m(\tau)(E-V(\tau,y))}\,dy\cdot\exp\left\{2\int_0^{\tau}\frac{b(\tau)}{m(\tau)}\,d\tau\right\}=\text{const}.$$

Important for applications is the case of large energies. Let us write, for example, the energy equation for a function f of the special form: $f(\tau,y,\dot{y})=\varphi(y,\tau)\dot{y}^{\alpha}$, $-1<\alpha\leq 1$, for large values of E. Up to and including terms of order $1/E$, we find

$$\begin{aligned}\frac{dE}{d\tau}=&-\frac{\partial m}{\partial\tau}\frac{E}{m}+\frac{2^{\frac{\alpha-1}{2}}}{\pi}\left(\frac{E}{m}\right)^{\frac{\alpha+1}{2}}\int_0^{2\pi}\varphi\,dy+\frac{1}{2\pi m}\int_0^{2\pi}\frac{\partial mV}{\partial\tau}\,dy\\&-\frac{2^{\frac{\alpha-3}{2}}}{\pi}m^{-\frac{\alpha+1}{2}}E^{\frac{\alpha-1}{2}}\left[\frac{1}{2\pi}\int_0^{2\pi}V\,dy\int_0^{2\pi}\varphi\,dy+|\alpha|\int_0^{2\pi}\varphi V\,dy\right]\\&+\frac{1}{E}\left[\frac{1}{4\pi m}\frac{\partial m}{\partial\tau}\int_0^{2\pi}V^2\,dy-\frac{1}{8\pi^2 m}\frac{\partial m}{\partial\tau}\left(\int_0^{2\pi}V\,dy\right)^2+\frac{1}{4\pi}\int_0^{2\pi}V\frac{\partial V}{\partial\tau}\,dy\right.\\&\left.-\frac{1}{8\pi^2}\int_0^{2\pi}V\,dy\int_0^{2\pi}\frac{\partial V}{\partial\tau}\,dy\right]+\frac{2^{\frac{\alpha-7}{2}}}{\pi}m^{-\frac{\alpha+1}{2}}E^{\frac{\alpha-3}{2}}\left[\frac{|\alpha|}{\pi}\int_0^{2\pi}V\,dy\int_0^{2\pi}\varphi V\,dy\right.\\&\left.+|\alpha|(|\alpha|-2)\int_0^{2\pi}\varphi V^2\,dy+\frac{1}{2\pi^2}\left(\int_0^{2\pi}V\,dy\right)^2\int_0^{2\pi}\varphi\,dy-\frac{3}{2\pi}\int_0^{2\pi}V^2\,dy\int_0^{2\pi}\varphi\,dy\right]+O\left(\frac{1}{E^2}\right).\end{aligned}$$

The asymptotics of rotational motions at large energies was considered in (6).

§ 4. **Physical examples.** As a first example, let us consider the Einstein pendulum (a mathematical pendulum whose length changes slowly and smoothly under the action of external forces) in the rotational regime. The equation of such a pendulum has the form

$$\frac{d}{dt}\left[x^2(\tau)\dot{y}\right] + gx(\tau)\sin y = 0, \qquad \tau = \varepsilon t, \tag{10}$$

where y is the angular displacement, $x(\tau)$ is the length of the string, and g is the acceleration of free fall. For the energy of the rotational motion of the pendulum, from the general equation (9) one obtains the expression $x(\tau)\sqrt{E}\,G\left(\sqrt{\frac{2gx(\tau)}{E}}\right) =$ const, where by G we have denoted the complete elliptic integral of the second kind. For large energies, up to accuracy $1/E^2$, we find the relation between the energy and the length of the pendulum:

$$E = \frac{x_0^2}{x^2}\left[E_0 + \frac{g}{x_0^2}(x^3 - x_0^3)\right] + \frac{g^2}{8x_0^2x^2E_0}(x^6 - x_0^6) + \bar{O}\left(\frac{1}{E^2}\right), \tag{11}$$

where x_0, E_0 are, respectively, the length and energy of the pendulum at the initial instant of time. Let now the length of the pendulum change slowly, but not under the action of external forces, rather at the expense of the system’ s own energy: the elastic string of the pendulum is stretched under the action of its weight and the centrifugal force. Such a pendulum is described by the equations:

$$\frac{d}{dt}\left[x^2\dot{y}\right] + gx\sin y = 0, \qquad \dot{x} = \varepsilon\lambda\left(mg\cos y + mx\dot{y}^2\right), \tag{12}$$

where $\varepsilon\lambda$ is a small coefficient of “plastic” deformation of the string. Oscillatory motions of system (12) were considered in $(^{4,5})$. Let us study rota-

motions of such a pendulum. Restricting ourselves to the case of large energies, we obtain from the general equations (7), to accuracy up to terms of order $1/E^2$, the expression

$$E = \frac{x_0^2}{x^2}\left[E^2 + \frac{g}{x_0^2}\left(x^3 - x_0^3\right)\right] - \frac{g^2}{4x_0^2x^2E_0}\left(x^6 - x_0^6\right) + O\left(\frac{1}{E^2}\right). \tag{13}$$

Comparing (11) and (13), we see that the energy in cases (10) and (12) changes in different ways, even if the lengths of both pendulums in the first approximation change in the same way.

Let us consider the example proposed in $^{(6)}$. The equation of motion (a generalization of the van der Pol equation) has the form

$$\ddot{y} + k^2\sin y = \varepsilon\left(a - b\sin^2 y\right)\dot{y},$$

where a, b, k are functions of the slow time $\tau = \varepsilon t$, and ε is a small parameter. The equation for the energy in the form (7) in this case is written as follows:

$$\frac{dE}{d\tau} = \frac{1}{2K\left(\sqrt{\dfrac{2k^2}{E}}\right)}\left[4aEG\left(\sqrt{\frac{2k^2}{E}}\right) - b\pi EF\left(\frac{3}{2}, -\frac{1}{2}, 3, \frac{2k^2}{E}\right) + \right.$$

$$\left. + 2\pi k\frac{dk}{d\tau}F\left(\frac{3}{2}, \frac{1}{2}, 2, \frac{2k^2}{E}\right)\right]. \tag{14}$$

Here G is the complete elliptic integral of the second kind, K is the complete elliptic integral of the first kind, and F is the hypergeometric function. If $k = \text{const}$, then the last term on the right-hand side vanishes.

For the case of large energies considered in $^{(6)}$, neglecting in equation (14) terms of order $1/E$ and integrating, we obtain

$$E(\tau) = \exp\left\{\int_0^\tau (2a-b)\,d\tau\right\} \times$$

$$\times\left[E_0 + \int_0^\tau \left(2k\frac{dk}{d\tau} - (2a-b)k^2\right)\exp\left\{-\int_0^\tau (2a-b)\,d\tau\right\}d\tau\right].$$

For $a = \text{const},\ b = \text{const},\ k = \text{const}$,

$$E(\tau) = E_0\exp\{(2a-b)\tau\} + k^2\left(1 - \exp\{(2a-b)\tau\}\right).$$

In conclusion, we note that by this method one can compute higher approximations, and also consider cases in which the right-hand side of the perturbed equation explicitly depends periodically on the time t.

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