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Abstract

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ON THE SPECTRUM OF NON-SELF-ADJOINT DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS

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At the All-Union Conference on Differential Equations in Kharkov (1957), I. M. Gel' fand posed the question of the topology of the spectrum of an equation with complex periodic coefficients. Some results in this direction are given below.

Consider the closed differential operator L, generated in the space $\mathcal{L}^2(-\infty, \infty)$ by the differential expression

$$l[y] = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y \tag{1}$$

with periodic complex coefficients

$$p_j(x+\omega) = p_j(x), \quad j = 0, 1, \dots, n, \quad \omega > 0, \quad p_0(x) \neq 0, \tag{2}$$

smooth to the extent required. Denote by $y_1(x, \lambda), \dots, y_n(x, \lambda)$ the fundamental system of solutions of the equation

$$l[y] = \lambda y \quad (-\infty < x < \infty) \tag{3}$$

under the initial conditions

$$y_j^{(k)}(0, \lambda) = \delta_{j,k+1} \quad (j, k+1 = 1, 2, \dots, n). \tag{4}$$

Under a shift by one period this system passes into a new fundamental system of solutions

$$y_j(x+\omega) = \sum_{k=1}^{n} a_{kj} y_k(x),$$

where the matrix $A(\lambda) = \|a_{kj}\|$, the so-called monodromy matrix of equation (3), coincides with the Wronski matrix for the system of solutions (4) at $x = \omega$: $a_{kj} = y_j^{(k-1)}(\omega, \lambda)$; $k, j = 1, \dots, n$.

The characteristic numbers $\rho_1(\lambda), \dots, \rho_n(\lambda)$ of the matrix $A(\lambda)$—the roots of the equation $\det|A(\lambda) - \rho \cdot I| = 0$—are called the multipliers of equation (3).

Definition 1. The set of conditional stability Λ of equation (3) is the set of those values λ for which this equation has at least one nontrivial solution bounded on the entire axis $-\infty < x < \infty$.

It follows from Floquet theory that if $\lambda \in \Lambda$, then at least one of the multipliers of equation (3) has modulus one, and conversely. Therefore Λ is determined by the characteristic equation with $\rho = e^{it}$, where t ranges over the interval $[-\pi, \pi]$:

$$\det|A(\lambda) - e^{it} I| = 0, \quad -\pi \leq t \leq \pi. \tag{5}$$

I. M. Gel' fand [(1)] has a method for constructing Parseval' s formula for equation (3) in the self-adjoint case, based on the use of solutions of equation (3) for values of λ determined by equation (5). From this construction one can conclude that the spectrum of the operator L in the self-adjoint case coincides with the set Λ. We shall show that the same is true also in the general non-self-adjoint case.

Theorem 1. The spectrum of the operator L with periodic complex-valued coefficients (1), (2) $(-\infty < x < \infty)$ is purely continuous, coincides with the set of conditional stability Λ, and consequently either consists of an infinite (in general) number of analytic arcs determined by equation (5), or fills the whole plane (if there is a multiplier independent of λ with unit modulus), or is completely absent (if the multipliers are constant and their moduli are $\neq 1$).

A sufficient condition excluding the last two possibilities is

$$p_0(x) = \text{const}^*.$$

Proof. Constructing a sequence of "almost eigenfunctions" for $\lambda \in \Lambda$, we conclude, on the basis of Weyl' s criterion, that λ is a point of the continuous spectrum (the operator L can have no eigenvalues and residual spectrum). Conversely, if $\lambda \notin \Lambda$, then λ is a regular point of the operator L, as is shown by the following lemma.

Lemma 1. If $\lambda \notin \Lambda$, then the operator

$$R_\lambda = (L - \lambda I)^{-1}$$

is a bounded integral operator defined on all of $\mathcal{L}^2(-\infty, \infty)$ with a Carleman-type kernel $G(x, \xi, \lambda)$, and

$$G(x + \omega, \xi + \omega, \lambda) = G(x, \xi, \lambda)$$

and

$$|G(x, \xi, \lambda)| \leq C e^{-\alpha|x-\xi|} (1 + |x - \xi|)^{m-1},$$

where

$$C = C(\lambda) > 0; \qquad \alpha = \frac{1}{\omega} \min_j |\ln|\rho_j(\lambda)|| > 0; \qquad m = m(\lambda) \leq n$$

is the largest exponent of the elementary divisors of the monodromy matrix.

Let us note that the kernel $G(x, \xi, \lambda)$ may also turn out to be triangular.

The proof of the lemma is carried out by a method generalizing the construction recently proposed by M. Burnat ([3]) for the case of a self-adjoint operator of the second order.

Finally, the impossibility, for $p_0(x) = \text{const}$, of the cases indicated in the theorem in which Λ is empty or fills the whole plane follows from the known asymptotics for the solutions of equation (3) as $|\lambda| \to \infty$.

Corollary. For all λ from a connected component of the resolvent set of the operator L, the number of multipliers with modulus < 1 is constant.

Example 1. The continuous spectrum of the problem

$$e^{-2ix}(y'' - iy') + \lambda y = 0 \qquad (-\infty < x < \infty) \tag{6}$$

fills the whole plane, since $A(\lambda) \equiv I$ for $\omega = 2\pi$. Here a fundamental system of solutions is

$$\operatorname{ch}[\sqrt{\lambda}(e^{ix} - 1)]; \quad (i\sqrt{\lambda})^{-1} \operatorname{sh}[\sqrt{\lambda}(e^{ix} - 1)].$$

Example 2. The problem obtained from (6) by the substitution $z = e^{-x}y$ has no spectrum at all:

$$e^{-2ix}[z'' + (2 - i)z' + (1 - i)z] + \lambda z = 0 \qquad (-\infty < x < \infty). \tag{7}$$

Example 3. This example concerns problems on the half-axis. Equation (7) with the boundary condition $z(0) = 0$, considered on the half-axis $0 \leq x < +\infty$, has every point of the λ-plane as an eigenvalue. For the same problem on the half-axis $-\infty < x \leq 0$, the entire λ-plane is filled with residual spectrum.

Definition 2. The set of ***s*-fold stability** Λ_s of equation (3) is the collection of those values λ for which this equation has exactly s linearly independent solutions of no more than polynomial growth as $x \to \pm\infty$. (In fact, this is an s-fold spectrum.)

Obviously,

$$\Lambda = \bigcup_{s=1}^{n} \Lambda_s, \qquad \Lambda_s \subset \Lambda.$$

The sets

$$\bigcup_{s=k}^{n} \Lambda_s$$

are closed ($k = 1, \dots, n$).

* In the survey ([2]), pp. 54 and 59, there is a reference to some unpublished results of McGarvey on the spect

Let us dwell in more detail on the spectrum of the second-order problem with complex coefficients of period $\omega > 0$:

$$-y'' + p_1(x)y' + p_2(x)y = \lambda y \qquad (-\infty < x < \infty). \tag{8}$$

A number of topological and metric properties of the set of conditional stability of this problem were established in the works of M. I. Serov ([5]). From Theorem 1 it follows directly that the same results are also valid for the spectrum of problem (8). Thus, if $\operatorname{Re} P = 0$, where

$$P = \int_0^\omega p_1(x)\,dx, \tag{9}$$

then $\Lambda = \Lambda_2$; the spectrum consists of a finite or infinite number of arcs contained in the strip $|\operatorname{Im}\lambda| < \text{const}$, and extending to the right to infinity.

If, however, $\operatorname{Re} P \neq 0$, then $\Lambda = \Lambda_1$. The spectrum contains one unbounded component, asymptotically approaching the parabola

$$\lambda = (t-\gamma)^2 + C \qquad (-\infty < t < \infty), \quad \gamma = \frac{1}{2i}P \tag{10}$$

(here C is a complex constant depending on the coefficients of the equation), and, possibly, several simple closed contours. No contour* can be contained inside another (for the infinite branch of the spectrum, the interior is taken to mean the domain containing $+\infty$). No two contours have more than one common point, and if such a point exists, it is a corner point for them.

Theorem 2. *The resolvent set of problem (8) is connected if and only if*

$$\operatorname{Re} P = 0.$$

Proof is required only in one direction. Let $\operatorname{Re} P = 0$, i.e. $\operatorname{Im}\gamma = 0$. Equation (5), which determines the spectrum, is reduced to the form

$$\Psi(\lambda) = [y_1(\omega,\lambda) + y_2'(\omega,\lambda)]e^{-i\gamma} = 2\cos(t-\gamma) \qquad (-\pi \le t \le \pi), \tag{11}$$

and on the spectrum we have $\operatorname{Im}\Psi(\lambda) = 0$. Therefore the spectrum cannot contain closed contours, since $\Psi(\lambda)$ is an entire function $\neq$ const. The spectrum also cannot contain two distinct solid branches going to infinity, as is seen from the asymptotics of $\Psi(\lambda)$ as $|\lambda| \to \infty$. Hence the resolvent set is connected when $\operatorname{Re} P = 0$.

Theorem 3. *The problem on the half-line $0 \le x < \infty$ for equation (8) with boundary condition $y'(0) - hy(0) = 0$ or $y(0) = 0$ has the same continuous spectrum as problem (8) on the whole axis. Moreover, if $\operatorname{Re} P < 0$, then the interior components of the resolvent set of problem (8) (including the one containing $+\infty$) are filled entirely with eigenvalues, a finite number of which may also lie on top of the continuous spectrum; and if $\operatorname{Re} P > 0$, then the same sets are filled with residual spectrum**. In addition, only isolated eigenvalues and points of the residual spectrum are possible, of which, when $\operatorname{Re} P \ne 0$, there are no more than finitely many. When $\operatorname{Re} P = 0$, the discrete and residual spectrum cannot be superimposed on the continuous spectrum.*

* The boundedness requirement for the contours considered, contained in (5), is superfluous.

** From these sets one may, in particular, single out an infinite sequence of points to which correspond the solutions of the boundary-value problem belonging to a single multiplier (and not linear combinations of solutions with different multipliers).

Theorem 4. The set of n-fold stability Λ_n of the J-self-adjoint differential equation of order $n = 2m$

$$y^{(2m)} + \left[p_1(x) y^{(m-1)}\right]^{(m-1)} + \cdots + p_m(x) y = \lambda y \tag{12}$$

with periodic complex-valued coefficients (2) contains no closed contours in the λ-plane. The spectrum of problem (12) ($-\infty < x < \infty$) consists of sets of even-fold stability.

Proof. The proof is based on the fact that, on the one hand, the sum of the multipliers is an entire function of λ, distinct from a constant (the trace of the monodromy matrix), while, on the other hand, for $\lambda \in \Lambda_n$ this sum is a real quantity, since, by the Lyapunov–Poincaré theorem ((4), p. 293), together with each multiplier ρ, the equation (12) also has the multiplier ρ^{-1} of the same multiplicity.

We note that the spectrum (12) may, in general, contain closed contours.

Example 4. The spectrum of the problem

$$y^{\mathrm{VI}} + (1+i) y^{\mathrm{IV}} + i y'' = \lambda y \quad (-\infty < x < \infty),$$

by virtue of the theorem on the mapping of spectra, forms the curve ($\lambda = \sigma + i\tau$):

$$\sigma = -t^4(t^2 - 1), \quad \tau = t^2(t^2 - 1) \quad (-\infty < t < \infty),$$

which contains a loop in the fourth quadrant.

Remark. The results of the present work are easily transferred to systems of differential equations with periodic coefficients.

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Note: Figure translations are in progress. See original paper for figures.