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Abstract

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ON RIEMANNIAN SPACES OF THE FIRST THREE LACUNARITIES IN THE GEOMETRIC SENSE

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In this note we determine the Riemannian spaces of the first three lacunarities that are maximally mobile in the sense of homothetic motions, under the assumption of a general metric of the space (it may be properly Riemannian or pseudo-Riemannian). A point transformation of a Riemannian space V_n is called **homothetic** if it carries the fundamental tensor $g_{\alpha\beta}(x^1, x^n)$ into itself up to a certain constant multiplier λ (the homothety coefficient), depending on the transformation. A homothetic transformation corresponding to the value of the homothety coefficient $\lambda = 1$ is called **trivial**. The totality of all homothetic transformations of the space V_n forms the group of homothetic motions $\mathfrak{G}_z$. In what follows we are concerned with groups containing nontrivial homothetic transformations.

The vector fields $\xi^\alpha(x^1, x^2, \dots, x^n)$ of infinitesimal homothetic motions of a Riemannian space V_n are determined by the following system of partial differential equations:

$$Dg_{\alpha\beta} = 2Cg_{\alpha\beta}, \tag{1}$$

where $g_{\alpha\beta}(x)$ is the metric tensor of the space V_n, C is a certain constant, and the left-hand side denotes the Lie derivative of the metric tensor along the field $\xi^\alpha(x^1, \dots, x^n)$.

The number r of fundamental solutions (linearly independent with constant coefficients) of system (1) characterizes the order of the group $\mathfrak{G}_r$. The group $\mathfrak{G}_r$ of homothetic motions contains, as a subgroup, the group of motions $\mathfrak{G}$ of order $r-1$, and $\mathfrak{G}$ is a $(1,2)$ normal divisor of the group $\mathfrak{G}_r$. Consequently, the order r of the group of homothetic motions of Riemannian spaces V_n is not greater than $n(n+1)/2+1$. Further, from the previously obtained results [3,4] on the lacunar distribution of the orders of groups of motions there follows a lacunar distribution of the orders of groups of homothetic motions.

In what follows we rely essentially on the results of [3,4]; spaces of the first lacunarity in the sense of isometric motions are spaces of constant curvature. In other words, Riemannian spaces V_n admitting groups of motions of order

$$r > n(n-1)/2 + 1$$

must admit the full group of motions of order

$$r = n(n+1)/2.$$

Further, spaces of the second lacunarity (i.e., spaces with a group of motions $\mathfrak{G}_r$, whose order r satisfies the inequalities

$$(n-1)(n-2)/2 + 5 < r \le n(n-1)/2 + 1$$

) are subprojective spaces, and the order of the full groups of motions of these spaces is equal to $n(n-1)/2$ or $n(n-1)/2 + 1$.

Thus, Riemannian spaces V_n admitting groups of motions $\mathfrak{G}$ of order

$$r > (n-1)(n-2)/2 + 5$$

are either spaces of the first lacunarity, and then for the full groups of motions

$$r = n(n+1)/2,$$

or spaces of the second lacunarity, and then for the full groups of moti-

motions $r = n(n-1)/2$ or $r = n(n-1)/2 + 1$. Further, the maximally mobile spaces of the third lacunarity (4) are determined by the line element

$$\begin{aligned} ds^2 = dx^{1\,2} + dx^{2\,2} - dx^{3\,2} - dx^{4\,2} + e_5 dx^{5\,2} + \dots + e_n dx^{n\,2} \\ + \begin{vmatrix} x^2 + x^3 & x^4 - x^1 \\ dx^2 + dx^3 & dx^4 - dx^1 \end{vmatrix}^2, \end{aligned} \tag{2}$$

where $e_a = \pm 1$ $(a = 5, \dots, n)$. The space (2) admits a group of homothetic motions $\mathfrak{G}_r$ of order $r = (n-1)(n-2)/2 + 6$. The operators of this group are

$$X_1 = p_2 - p_3, \quad X_3 = p_2 + (x^4 - x^1)(x^2 + x^3)(p_1 + p_4) + (x^4 - x^1)^2(p_2 - p_3),$$
$$X_2 = p_1 + p_4, \quad X_4 = p_4 - [(x^2 + x^3)^2(p_1 + p_4) + (x^4 - x^1)(x^2 + x^3)(p_2 - p_3)],$$
$$X_5 = (x^2 + x^3)(p_1 + p_4) + (x^4 - x^1)(p_2 - p_3),$$
$$X_6 = x^4 p_1 + x^3 p_2 + x^2 p_3 + x^2 p_4,$$
$$X_7 = (x^2 + x^3)(p_4 - p_1) + (x^4 + x^1)(p_2 - p_3),$$
$$X_8 = (x^3 - x^2)(p_4 + p_1) + (x^1 - x^4)(p_2 + p_3),$$
$$X_{0a} = p_a, \quad X_{ab} = e_a x^a p_b - e_b x^b p_a \quad (\text{do not sum over } a, b!),$$
$$Y_{1a} = e_a x^a (p_3 - p_2) + (x^2 + x^3) p_a, \quad Y_{2a} = e_a x^a (p_4 + p_1) + (x^4 - x^1) p_a,$$
$$X = x^4 p_1 - x^3 p_2 + x^2 p_3 + x^1 p_4 + 2 \sum_{a=1}^{n} x^a p_a \quad \left(p_\alpha = \frac{\partial}{\partial x^\alpha} \right). \tag{3}$$

The following facts hold concerning lacunae and the maximally mobile spaces connected with them.

Theorem 1. A Riemannian space V_n admits a group of homothetic motions $\mathfrak{G}_r$ of maximal order $r = n(n+1)/2 + 1$ if and only if it is flat (spaces of the first lacunarity).

Theorem 2. There do not exist Riemannian spaces V_n admitting full groups of homothetic motions $\mathfrak{G}_r$, if the order r satisfies the inequality

$$n(n-1)/2 + 2 < r < n(n+1)/2 + 1.$$

We next introduce, for the determination of homothetic spaces of the second lacunarity, a conformally Cartesian coordinate system. The subprojective spaces of the basic case, defined in a proper coordinate system by the metric

$$ds^2 = \varphi(x^1)\,[e_1 dx^{1\,2} + e_2 dx^{2\,2} + \cdots + e_n dx^{n\,2}], \tag{4}$$

are homothetic spaces if and only if $\varphi(x^1)$ is equal to e^{ax^1} or $(ax^1 + b)^c$; here a, b, c are constants, with $a \neq 0$, $b \neq 0$, $c \neq -2$. The operator of homothetic motions corresponding to these cases can be brought to the form

$$Xf = \frac{\partial f}{\partial x^1}, \qquad Xf = b \frac{\partial f}{\partial x^1} + a x^\sigma \frac{\partial f}{\partial x^\sigma}.$$

The remaining subprojective spaces of the basic case have semisimple groups of motions or do not admit nontrivial homothetic motions, or else admit convergent (concurrent) homotheties.

It is then not difficult to verify that the subprojective spaces of the exceptional case admit nontrivial homothetic motions; the metric of these spaces,

$$ds^2 = \varphi(x^1)\,[2dx^1dx^2 + e_3dx^{3\,2} + \cdots + e_n dx^{n\,2}], \tag{5}$$

conformal to a flat space, also depends on one arbitrary function $\varphi(x^1)$. The subprojective spaces admitting groups of homothetic motions $\mathfrak{G}_r$ of order $r = (n-1)n/2 + 2$ are obtained from (5)

for $\varphi(x^1) = e^{2x^1},\ (x^1)^a,\ \dfrac{e^{-2a\,\mathrm{arctg}\,x^1}}{1 + x^{1^2}}$. The homothety operator is

$$Xf = 2x^2\frac{\partial f}{\partial x^2} + x^3\frac{\partial f}{\partial x^3} + \cdots + x^n\frac{\partial f}{\partial x^n}.$$

Thus, we obtain the following propositions on the second lacuna and on the spaces of second lacunarity associated with it.

Theorem 3. *The Riemannian spaces V_n of second lacunarity in the homothetic sense are subprojective spaces; they admit groups of homothetic motions $\mathfrak{G}_r$ of orders $n(n-1)/2+1,\ n(n-1)/2+2$. All subprojective spaces of the exceptional case possess homothetic motions.*

Theorem 4. *There do not exist Riemannian spaces V_n admitting groups of homothetic motions $\mathfrak{G}_r$, whose order r satisfies the relation $(n-1)(n-2)/2+6 < r < n(n-1)/2+1$.*

The space of the third lacunarity maximally mobile in the homothetic sense is the space (2); it admits the group (3) of order

$$r = (n-1)(n-2)/2 + 6.$$

Theorem 5. *The maximal order r of groups $\mathfrak{G}_r$ of homothetic motions of nonconformally Euclidean spaces is exactly $(n-1)(n-2)/2+6$.*

Theorem 6. *The group $\mathfrak{G}_r$ of homothetic motions of a Riemannian space V_n necessarily coincides with the group of motions $\mathfrak{G}$ if V_n is of nonzero constant scalar curvature.*

This theorem is a generalization of Yano’ s theorem (5) on the coincidence of $\mathfrak{G}_r$ and $\mathfrak{G}$ for Einstein V_n possessing a nonzero Ricci tensor.

In the case $n = 2$ we have the following interesting proposition.

Theorem 7. *Spaces V_2 of nonzero curvature that are maximally mobile in the homothetic sense possess two-parameter groups of homothetic motions; V_2 of nonzero curvature ($K \neq 0$) admit a complete group $\mathfrak{G}_2$ of homothetic motions if and only if it is mapped onto a surface of revolution, and moreover*

$$\Delta_1 K = aK\Delta_2 K = bK^3,$$

where Δ_1 and Δ_2 denote the first and second differential parameters, and a, b are constants.

It is not difficult to show that the line element of the V_2 under consideration can, in a special coordinate system, be reduced to the form

$$ds^2 = dx^{1^2} + e\,\frac{dx^{2^2}}{x^{1^{2t}}},$$

where $e = \pm 1$. The operators of the group $\mathfrak{G}_2$ of homothetic motions are

$$X_1 f = \frac{\partial f}{\partial x^2}, \qquad X_2 f = x^1 \frac{\partial f}{\partial x^1} + (t+1)x^2 \frac{\partial f}{\partial x^2}.$$

The structure of the group is $[X_1, X_2] = (t+1)X_1$, where the constant $t \neq -1$.

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Note: Figure translations are in progress. See original paper for figures.