A. Arhangel' skii

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Abstract

Full Text

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SOME TYPES OF QUOTIENT MAPS AND RELATIONS BETWEEN CLASSES OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 18 VI 1963)

All spaces are assumed to be Hausdorff, all maps continuous and single-valued.

§ 1. **Definition 1.** A map $f : X \to Y$ of a topological space X onto a topological space Y is called a **quotient** map (or a **factor map**) if a set $V \subseteq Y$ is open if and only if the set $U = f^{-1}V$ is open in X (in this case we shall sometimes call Y a **quotient space** of the space X)*.

Obviously, quotient maps include both open and closed maps; however, as is well known, the set of quotient maps is not exhausted by these classes of maps.

Consider a map $f : X \to Y$, and let $Y_1 \subseteq Y$ be an arbitrary subspace of the space Y and $X_1 = f^{-1}Y_1$. The map f induces a map $f_1 : X_1 \to Y_1$. It is well known that if f has one of the properties: continuous, open, or closed, then the map $f_1 : X_1 \to Y_1$ will also have the same property. It is precisely this circumstance that we shall have in mind when saying that a certain class of maps is hereditary.

It requires no explanation that the convenience of the class of maps under consideration depends in many respects on the presence of the heredity property, and this is one of the first requirements that should be imposed on every good class of maps.

It turns out that the class of quotient maps is not hereditary. The question arises: what is the minimal narrowing of this class that satisfies the heredity requirement? In other words, what are hereditary quotient maps?

Definition 2. A map $f : X \to Y$ is called **pseudo-open** if, for every point $y \in Y$ and every set $U \supset f^{-1}y$ open in X, the relation $\operatorname{Int} fU \ni y$ holds.

Proposition 1. *Every pseudo-open map is hereditarily pseudo-open.*

Proposition 2. *Every pseudo-open map is a quotient map.*

Sometimes the following property of pseudo-open maps proves very useful:

Proposition 3. *In order that a map $f : X \to Y$ be pseudo-open, it is necessary that, for an arbitrary open covering $\gamma = \{U_\alpha \mid \alpha \in M\}$ of the space X, the system $\varphi = \{sf\gamma = \{\operatorname{Int}\operatorname{star} f\gamma(y) \mid y \in Y\}$, consisting of the cores of the stars of points of Y with respect to the system of all images of elements of γ, cover Y.*

Theorem 1. *A map $f : X \to Y$ is pseudo-open if and only if it is hereditarily quotient.*

§ 2. Pseudo-open maps make it possible to establish arbitrary relations between classes of topological spaces.

* Quotient maps, under many other names (quasicompact map, quotient map, quasi-open), have been considered for a long time (see the book ([6])). They were first introduced by P. S. Aleksandrov in ([6]).

Theorem 2. Fréchet spaces* and only they are the pseudo-open images of metric spaces.

Theorem 3. k'-Π-spaces and only they are the pseudo-open images of locally bicompact spaces (the definition of a k'-space is given in ([2])).

Theorem 2 supplements the following criterion, very essential for us, for when a quotient space of a Fréchet space is a Fréchet space.

Theorem 4. Let $f : X \to Y$ be a quotient mapping of a Fréchet space X onto a topological space Y. Then Y is a Fréchet space if and only if f is pseudo-open.

§ 3. An extraordinarily difficult problem is to find a criterion for when the image of a metrizable space under a continuous mapping is metrizable, in terms of restrictions imposed on the mapping. In such a formulation, for intuitively clear reasons, the very existence of any substantive solution seems almost incredible.

Stone ([5]) found a criterion for when the image of a metrizable space under a closed mapping is metrizable—the necessary and sufficient condition turned out to be the peripheral bicompactness of the mapping f. (In part this result belongs to I. A. Vainshtein ([3]).)

The purpose of this paragraph is to give a necessary and sufficient condition for a quotient space of a metrizable space to be metrizable, in terms of properties of the corresponding quotient mapping.

First we introduce some notation. Let $f : X \to Y$ be a mapping and let $U \subseteq X$ be a set open in X. Put $A(U) = f^{-1}fU$; U is called a **marked set** if $A(U) = U$. A set F is an element of the decomposition if, for some point $y \in Y$, $F = f^{-1}y$.

Theorem 5. Let $f : X \to Y$ be a quotient mapping of a metrizable space X. In order that the quotient space Y** be metrizable, it is necessary and sufficient that there exist on X a metric ρ satisfying the following condition: for every element of the decomposition $F \subseteq X$ and every neighborhood $U \supseteq F$: 1) there is a neighborhood $V \supseteq F$ such that $\rho(X \setminus A(U), A(V)) > 0$; 2) there is an open marked set G for which $F \subseteq G \subseteq A(U)$.

With the aid of the notion of a regular mapping of a metrizable space introduced in ([1]), and the notion of a pseudo-open mapping just introduced, the following theorem, essentially analogous to Theorem 5, can be formulated.

Theorem 5′. In order that the quotient space of a metrizable space X be metrizable, it is necessary and sufficient that the corresponding quotient mapping be regular and pseudo-open.

However, of these two closely related theorems, preference apparently should be given to the first (Theorem 5), whose formulation contains no assumptions on the topology of the space Y, except that it is a quotient space. (The requirement of regularity of the mapping does not formally satisfy this condition.)

§ 4. **Closed mappings.** This paragraph is only slightly connected with the preceding material. It contains information on how the following are arranged: 1) closed mappings of locally bicompact spaces and 2) closed mappings of metric spaces. What is characteristic in the formulation of questions in both cases is that no a priori restrictions are imposed on the image space.

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* A topological space X is called a **Fréchet space** if from $x \in [M]$ it follows that there exists a sequence $\{x_n\}$, $x_n \in M$, converging to x.

** The space Y is not assumed in advance even to be Hausdorff.

A. **Theorem 6.** Let $f : X \to Y$ be a closed mapping of a metric space X. Then $Y = Y' \cup Y''$, where

$$Y'' = \bigcup_{i=1}^{\infty} Y_i,$$

where each Y_i is a discrete subset of Y, and for an arbitrary point $y \in Y'$ the set $f^{-1}y$ is totally bounded in the metric of X (and hence has a countable base).

Corollary 1. If $f : X \to Y$ is a closed mapping of a complete metric space X, then $Y = Y' \cup Y''$, where

$$Y'' = \bigcup_{i=1}^{\infty} Y_i,$$

where all Y_i are discrete in Y, and for every point $y \in Y'$ the set $f^{-1}y$ is bicompact and, consequently, the space Y' is metrizable.

Corollary 2. The closed image of a completely metrizable space is the sum of at most a countable set of metrizable spaces.

B. It is well known that the image of a locally bicompact space under a closed mapping need not be locally bicompact.

Theorem 7. Let $f : X \to Y$ be a closed mapping of a locally bicompact weakly paracompact space X.

Then $Y = Y' \cup Y''$, where Y'' is discrete in Y, and for every point $y \in Y'$ the set $f^{-1}y$ is bicompact.

Corollary. The closed image of a locally bicompact weakly paracompact space is always the sum of at most two locally bicompact spaces.

B. **The countable case.**

Theorem 8. Let $f : X \to Y$ be a closed mapping of a complete metric space with a countable base X. Then for all points $y \in Y$, with the possible exception of some countable set of them, $f^{-1}y$ is bicompact.

Theorem 9. The closed image of a metrizable locally bicompact space decomposes into the sum of two locally bicompact metrizable spaces, one of which is discrete in the image, and on the full preimage of the other the mapping under consideration is perfect.

If the original space has a countable base, then the first of these summands is countable.

G. The problem of finding closed images of metric spaces with a countable base is in an obvious way reduced to the problem of characterizing closed images of subspaces of the line. We give here a related partial result.

Theorem 10. Let f be a continuous mapping of the line R inducing a closed mapping f_1 on some subspace $M \subseteq R$. Then f_1 generates on M a decomposition all of whose elements, with the exception of at most a countable set, are bicompact.

Remark. Theorem 10 remains true if R is an arbitrary metric space of type F_σ, i.e. the sum of a countable set of bicompacts.

D. **Open-closed mappings.** A well-known result of Stone ([5]) on open-closed mappings of metric spaces is strengthened by

Theorem 11. If $f : X \to Y$ is an open-closed mapping of a k-space X that is complete in the sense of Dieudonné*, then $Y = Y' \cup Y''$, where all points of Y'' are isolated in Y, and for every point $y \in Y'$ its full preimage $f^{-1}y$ is bicompact.

Corollary 1. If $f : X \to Y$ is an open-closed mapping of a k-space X, complete in the sense of Dieudonné, onto a space Y without isolated points, then f is perfect.

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* That is, complete with respect to the maximal uniform structure compatible with the topology of the space.

With the aid of V. Ponomarev' s theorem, from (4), Theorem 11 yields

Theorem 12. *The image of a strongly paracompact k-space under an open-closed mapping is again a strongly paracompact space.*

Unsolved problem: to construct an example of a closed mapping of a metric space with a countable base such that an uncountable set of elements of the decomposition is not bicompact.

Note added in proof. The definition of a pseudo-open mapping under the name of preclosed mapping was given independently of me by Yu. M. Smirnov. As it turned out, the work of Dinh Nhe Tong is devoted to its study (7). Our main results do not overlap.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.