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Abstract

Full Text

PHYSICS

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ON THE ENTROPIC EXPRESSION OF THE UNCERTAINTY PRINCIPLE

(Presented by Academician I. V. Obreimov, 28 VI 1962)

For a quantitative expression of the uncertainty principle, the variances (or standard deviations) of the canonically conjugate quantities under consideration are usually chosen as a measure of uncertainty. Let us first turn to the one-dimensional case and consider a pair of quantities: the coordinate x and momentum p of a particle. Then the uncertainty principle is expressed by the Weyl inequality (^{1})

$$D_{xD}p \geq {}^1/4 \quad \text{(at. units)}, \tag{1}$$

where D_x, D_p are the variances of the coordinate and momentum, respectively. However, the variance characteristic of the uncertainties under consideration is not the only possible quantitative characteristic of them. It will be shown below that there exists another, stronger characteristic.

In a number of recent works ($^{2-5}$), the uncertainties of the position and momentum of a particle are defined by the entropy functionals H_x and H_p of the corresponding distributions, namely as follows*

$$H_x = -\int_{-\infty}^{\infty} |\varphi(x)|^2 \ln |\varphi(x)|^2 \, dx, \qquad H_p = -\int_{-\infty}^{\infty} |\psi(p)|^2 \ln |\psi(p)|^2 \, dp, \tag{2}$$

where $\varphi(x)$ is the state of the particle in the x-representation; $\psi(p)$ is the same state in the p-representation

$$\psi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-ipx} \, dx. \tag{3}$$

It is evident that the entropic expression of the uncertainty principle must have the form

$$H_x + H_p \geq C \qquad (C > -\infty, \text{ constant}). \tag{4}$$

Since, however, the values of the functionals D and H in the general case do not determine one another uniquely and are only related by the inequality

$$H \leq 0.5 \ln(2\pi e D), \tag{5}$$

with equality holding only for Gaussian distributions, the question arises of the relation between the two mathematical expressions of the uncertainty principle —the variance and the entropic. In this connection it is essential to note the logical dependence of the two expressions of the uncertainty principle at first in general form: from an inequality of the form (4), by virtue of (5), there follows an inequality of the form

$$D_{xD}p \geq C_1 \qquad (C_1 = e^{2C}/4\pi^2 e^2). \tag{6}$$

But from inequality (6) with the aid of (5) one cannot derive inequality (4). This means that the entropic expression of the uncertainty principle is, so to speak, "stronger in form" than its variance expression. However, vo- the question of the value of the constant C in (4) is of fundamental importance for establishing the formal-logical dependence between inequalities (4) and (6) in the case of a concrete assignment of the constant C_1, as is the case in inequality (1).

Hirschman (2) proved that for functions from $L_2(-\infty, \infty)$ for which the functional $H_x + H_p$ is defined, an entropic analogue of inequality (1) is indeed valid, namely,

$$H_x + H_p \geqslant \ln(2\pi) \qquad (\ln 2\pi \approx 1.838), \tag{7}$$

and he conjectured that the constant on the right-hand side of (7) can be improved and that the minimum of the functional $H_x + H_p$ is attained on Gaussian distributions, i.e., that, in general,

$$H_x + H_p \geqslant \ln \pi + 1 \qquad (\ln \pi + 1 \approx 2.145). \tag{8}$$

He noted that (1) can be derived from (8).

Let us now show that the entropic formulation of the uncertainty principle for canonically conjugate coordinate and momentum makes it possible, in general, to exclude the possibility of certain situations that are not excluded by Weyl’ s inequality by itself.

Indeed, suppose it is known that a particle is localized in sufficiently small neighborhoods of two or more (a finite number of) centers. This means that the

* Let us emphasize that the entropy of a separate quantum-mechanical distribution (for example, H_x or H_p) is a quantum-mechanical concept.

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Fig. 1

Figure 1: Fig. 1

entropy of the coordinate distribution is in any case less than zero (the closer the distribution is to a discrete one, the closer its entropy is to $-\infty$). The variance of such a distribution may be arbitrarily large, as is seen, for example, from Fig. 1.

Fig. 1

Weyl' s inequality, considered by itself, would in this case not exclude the possibility of determining the momentum with an arbitrarily high degree of accuracy, provided only that the variance for the coordinate is sufficiently large. At the same time it is obvious that the entropic form of the uncertainty principle in any case does not allow a negative value for the sum of the entropies of the momentum and coordinate distributions. This means that, under our assumption concerning the coordinate distribution, the entropy value for the momentum distribution can no longer in any case be negative, and therefore determination of the momentum value with a very high degree of accuracy is impossible.

The example given is fully consistent with the fact that, in order to justify the entropic form of the uncertainty principle, deeper facts from the theory of the Fourier transform are brought in than are required in deriving Weyl' s inequality. The example given also shows that the entropic form of the uncertainty principle makes it possible to discern certain general physical aspects of the matter that are not expressed by Weyl' s inequality.

We shall now show that the constant on the right-hand side of (7) can indeed be improved. Namely, the following is true.

Theorem 1. *The constant C in inequality (4) is in any case not less than* 2.100.

Proof. Let $\varphi(x) \in L_r(-\infty, \infty)$, and let $\psi(p)$ be defined by formula (3). Recently K. I. Babenko (6) showed that for even q related to r by the relation $q = r/(r-1)$, the inequality

$$\|\psi\|_q \leqslant M(r)\|\varphi\|_r, \tag{9}$$

holds, where

$$\|\varphi\|_k = \left\{\int_{-\infty}^{\infty} |\varphi(x)|^k \, dx\right\}^{1/k} \quad \text{and} \quad M(r) = \left(\frac{r}{2\pi}\right)^{1/2r} \left(\frac{q}{2\pi}\right)^{-1/2q}.$$

To the operator T,

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* It cannot be greater than $\ln\pi + 1$, since for every function whose squared modulus is the probability density of a Gaussian distribution, $H_x + H_p = \ln\pi + 1$.

defined by the formula $T\varphi = \dfrac{1}{\sqrt{2\pi}} \int \varphi(x) e^{-ipx}\,dx$, the following **Riesz theorem** (7) is applicable. Let $1/r_i + 1/q_i = 1$ $(i = 1, 2)$, $1 < r_i \leqslant 2$, $r_2 > r_1$, and let T be a linear bounded operator, defined both in $L_{r_1}(-\infty, \infty)$ and in $L_{r_2}(-\infty, \infty)$, such that $\|Tf\|_{q_i} \leqslant M_i \|f\|_{r_i}$. Then the operator T can be extended to $L_r(-\infty, \infty)$, where $r_1 < r < r_2$, and in this case

$$\|Tf\|_q \leqslant M_1^{(r-r_1)/(r_2-r_1)} M_2^{(r_2-r)/(r_2-r_1)} \|f\|_r.$$

Let us now set $r_1 = 4/3$, $q_1 = 4$; $r_2 = 2$, $q_2 = 2$. Then, according to Riesz' s theorem, for $r \in [4/3, 2]$ the inequality holds

$$\|\psi\|_q \leqslant M^{2(2/r-1)} \|\varphi\|_r, \tag{10}$$

where $M = 0.644^*$. Taking logarithms, we obtain

$$\frac{1}{q} \ln \int |\psi|^q\,dp \leqslant 2\left(\frac{2}{r} - 1\right) \ln M + \frac{1}{r} \ln \int |\varphi|^r\,dx. \tag{11}$$

Consider the function

$$\chi(r) = \frac{1}{q} \ln \int |\psi|^q\,dp - \frac{1}{r} \ln \int |\varphi|^r\,dx - 2\left(\frac{2}{r} - 1\right) \ln M.$$

Then from inequality (11) it follows that $\chi(r) \leqslant 0$ for $4/3 \leqslant r \leqslant 2$. Moreover, $\chi(2) = 0$. Therefore $d\chi(2-0)/dr \geqslant 0$. Formal differentiation of the function $\chi(r)$ leads to the expression**

$$\frac{d\chi(r)}{dr} = \frac{1}{r^2} \ln \int |\psi|^q\,dp - \frac{1}{r(r-1)} \frac{\int |\psi|^q \ln|\psi|\,dp}{\int |\psi|^q\,dp} + \frac{1}{r^2} \ln \int |\varphi|^r\,dx - \frac{1}{r} \frac{\int |\varphi|^r \ln|\varphi|\,dx}{\int |\varphi|^r\,dx} + \frac{4}{r} \ln M.$$

Taking into account that $\|\varphi\|_2 = \|\psi\|_2 = 1$, at $r = 2$ we obtain the inequality

$$-\int |\psi|^2 \ln|\psi|^2\,dp - \int |\varphi|^2 \ln|\varphi|^2\,dx + 4\ln M \geqslant 0.$$

Thus, the entropic expression of the uncertainty principle can be written at least in the form

$$H_x + H_p \geqslant 2.100. \tag{12}$$

It follows from this that

$$D_{xD}p \geqslant 0.228. \tag{13}$$

Thus, the values of the constants C and C_1 found in (12) and (13) differ from 2.145 and from 0.25, respectively, by 0.045 and 0.022.

The entropic expression of the uncertainty principle is also preserved for the multidimensional case. In this case the following holds:

Theorem 2. Let $f(x_1, x_2, \dots, x_n) = f(x)$, $\int |f(x)|^2 \, dx = 1$, $dx = dx_1 dx_2 \dots dx_n$,

$$g(p) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int f(x) e^{-i(p,x)} \, dx, \qquad (p,x) = \sum p_{ix} i.$$

Then

$$-\int |f|^2 \ln |f|^2 \, dx - \int |g|^2 \ln |g|^2 \, dp \geqslant n \ln 2\pi. \tag{14}$$

The proof of this theorem is carried out in the same way as in the one-dimensional case, using Riesz' s theorem, the generalization of which to the case of many variables was proved in (8). Apparently, in this case too inequality (14) is not sharp.

One can also, from a somewhat different point of view, discern the possibility of an inequality stronger than Weyl' s inequality. That inequality (1) imposes an excessively weakened condition on the distributions $|\varphi(x)|^2$ and $|\psi(p)|^2$ can also be seen clearly from the following considerations, connected with one form, not noted earlier, of the expression of the uncertainty principle.

Let $\varphi(x) \in L_2(-\infty, \infty)$ and let $\psi(p)$ be defined by formula (3); $g_\varphi(x)$ and $g_\psi(p)$ are the probability densities of two Gaussian distributions satisfying the conditions (see formulas (2))

$$-\int_{-\infty}^{\infty} g_\varphi(x) \ln g_\varphi(x) \, dx = H_x, \qquad -\int_{-\infty}^{\infty} g_\psi(p) \ln g_\psi(p) \, dp = H_p. \tag{15}$$

Denote by D_x^0 and D_p^0 the variances of the distributions $g_\varphi(x)$ and $g_\psi(p)$. Since for Gaussian distributions equality holds in formula (5), it is clear that

$$H_x = 0.5 \ln(2\pi e D_x^0), \qquad H_p = 0.5 \ln(2\pi e D_x^0). \tag{16}$$

Since, for a given value of the functional H, the Gaussian distribution has the minimal variance D, the variances D_x and D_p of the distributions $|\varphi(x)|^2$ and $|\psi(p)|^2$ can be written in the form $D_x = D_x^0 + D_x^*$ and $D_p = D_p^0 + D_p^*$. It is essential that if $|\varphi(x)|^2$ is not the density of a Gaussian distribution, then necessarily $D_x^* > 0$ and $D_p^* > 0$. From (4) and (16) it now follows that even for the product D_x^0 and D_p^0 the inequality

$$D_x^0 D_p^0 \geq C_1 \qquad (C_1 = e^{2C}/4\pi^2 e^2) \tag{17}$$

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* The numerical values of the quantities are given to an accuracy of 0.001.

** The operation of differentiation can be rigorously justified. See, in this connection, (2).

holds, with the same right-hand side as in (6) for the product $D_{xD}p$. It is clear that inequality (1) is substantially weakened by the "excess" terms D_x^* and D_p^* of the variances D_x and D_p in all cases, with the sole exception of the case when $|\varphi(x)|^2$ itself is the density of a Gaussian distribution.

Let us note that from an inequality of the form $D_x^0 D_p^0 \geq C_1$ there follows, conversely, an inequality of the form $H_x + H_p \geq C$, with C determined by the formula $C = 0.5 \ln C_1 + 0.5 \ln(4\pi^2 e^2)$. In particular, for $C_1 = 0.25$, $C = \ln \pi + 1$, i.e. the inequalities $H_x + H_p \geq \ln \pi e$ and $D_x^0 D_p^0 \geq 0.25$ are equivalent. For $C = 2.100$ we find $D_x^0 D_p^0 \geq 0.228$, whence, bearing in mind that for Gaussian distributions $D_{xD}p = 0.25$ (i.e. > 0.228), we obtain for the general case

$$D_{xD}p > 0.228. \tag{18}$$

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