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Abstract

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ON A GENERALIZATION OF REDUCTIVE HOMOGENEOUS SPACES

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P. K. Rashevskii (1) found a class of homogeneous spaces into which an invariant affine connection can be introduced. These spaces were later rediscovered by Nomizu (2) and called reductive. In the present work the concept of reductivity is generalized to such homogeneous spaces into which an affine connection obviously cannot be introduced. The result is as follows. For any $\nu = 1, 2, \ldots$ a certain class of homogeneous spaces R^ν and a class of groups D^ν are introduced, and it is proved that into a homogeneous space of class R^ν one can introduce an invariant connection with one of the groups D^ν. Here the class R^1 consists of reductive spaces, while the class D^1 contains only the general affine group and its subgroups. We also note that the class D^2 contains the projective and conformal groups.

We consider a local homogeneous space and shall specify it by its Lie algebra G and stationary subalgebra H.

1°. We shall say that the homogeneous space G/H is ν-reductive or belongs to the class R, if the stationary subalgebra H has a decomposition

$$H = H_1 \dot{+} H_2 \dot{+} \ldots \dot{+} H_\nu \quad (H_\nu \neq 0),$$

and in G there exists a plane E complementary to H such that

$$[H_i, E] \subset H_{i-1}, \quad i = 1, 2, \ldots, \nu \quad (H_0 \equiv E). \tag{1}$$

Obviously, for $\nu = 1$ we obtain reductive spaces.*

Lemma 1

$$[H_i, H_k] \subset H_{i+k-1} \quad (i = 1, 2, \ldots;\ k = 0, 1, 2, \ldots) \tag{2}$$

$$(H_l = 0,\ \text{if } l > \nu).$$

Lemma 2. If $a, b \in E$ and $h \in H_2 \dotplus \dots \dotplus H_\nu$, then $[h[a, b]] = 0$.

Let $A(u_1, u_2, \dots, u_k)$ and $B(u_1, u_2, \dots, u_l)$ be, respectively, k-linear and l-linear symmetric operators with values in an n-dimensional affine space T_0 ($u_1, u_2, \dots$ also belong to T_0), and let $A \equiv A(x, x, \dots, x)$, $B \equiv B(x, x, \dots, x)$ be the corresponding operators of orders k and l. Define the operator of order $k + l - 1$

$$A \circ B \equiv A(B(x, x, \dots, x), x, \dots, x). \tag{3}$$

It is convenient to regard vectors as 0-linear operators and to define

$$A \circ a \equiv A(a, x, \dots, x), \quad a \circ A \equiv 0. \tag{4}$$

We now assign to each element $h \in H_k$ $(k = 0, 1, \dots, \nu)$ the k-linear operator $[\dots[[h, u_1]u_2], \dots, u_k]$, acting on the plane E ($u_1, u_2, \dots, u_k \in E$), and shall denote

$$P_h \equiv [\dots[[h, x], x], \dots, x].$$

Lemma 3. The operator $[\dots[[h, u_1], u_2], \dots, u_k]$ is symmetric.

Theorem 1. Let $a \in H_k$ $(k = 1, 2, \dots, \nu)$, $b \in H_l$ $(l = 1, 2, \dots, \nu)$. Then

$$P_{[a,b]} = \frac{(k + l - 1)!}{k!\, l!}(kP_a \circ P_b - lP_b \circ P_a). \tag{5}$$

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* If $\nu > 1$, then $[H_i, E] \subset H$ for $i > 1$, whence it is seen that the linear representation of the stationary group in the tangent space is not faithful and, thus, an invariant affine connection cannot be introduced into the space R^ν.

2°. As is known, by a differential group of order ν one means the group of all ν-times continuously differentiable transformations of the coordinates $(x_1, x_2, \dots, x_n)$ of an n-dimensional manifold that leave fixed the origin O and are considered in a neighborhood of the origin O with accuracy up to order ν.

Consider the algebra $\mathfrak{A}$ (with respect to the operation of commutation) obtained from the Lie algebra of the differential group by adjoining the translation operators $(x^i \to x^i + a^i)$, in a fixed coordinate system. It is easy to see that $\mathfrak{A}$ is not a Lie algebra. We shall be interested in those subspaces of the algebra $\mathfrak{A}$ which form a Lie algebra and include all translation operators. We shall agree to call any subspace of this kind a Lie algebra of the family D^ν. If such a subspace is maximal, then we shall call it the Lie algebra of a transitive differential group of order ν.

Let us formulate these definitions in more precise terms. Denote by T_0 an n-dimensional vector space, by T_1 the space of all linear operators acting in T_0, by T_2 the space of all bilinear symmetric operators acting in T_0, etc., and by T_ν the space of ν-linear symmetric operators. To each k-linear symmetric operator $A(u_1, u_2, \dots, u_k)$ there corresponds one-to-one an operator of the k-th order $A(x, x, \dots, x)$. Let A and B be two operators of orders k and l. Define the commutator by the following formulas (where vectors of the space T_0 are regarded as operators of order 0):

$$[A, B] = \frac{(k+l-1)}{k!\,l!}(kA \circ B - lB \circ A), \qquad \text{if } 0 \le k+l-1 \le \nu, \tag{6}$$

$$[A, B] = 0, \qquad \text{if } k+l-1 > \nu \text{ or } k = 0,\ l = 0.$$

The direct sum $T_1 + T_2 + \dots + T_\nu$, in which the commutators are defined by formulas (6), forms the Lie algebra of the differential group of order ν. The direct sum $T_0 + T_1 + \dots + T_\nu$ does not form a Lie algebra (except in the case $\nu = 1$). We shall agree to call a Lie algebra belonging to the family D^ν if it is isomorphic to a linear subspace of the space $T_0 + T_1 + \dots + T_\nu$ that includes T_0. If this subspace is maximal, i.e. is not contained in any other subspace of this kind, we shall call it the Lie algebra of a transitive differential group of order ν.

To each Lie algebra G' of the family D^ν there corresponds uniquely a certain homogeneous space M', and the stationary subalgebra H' of the space M' consists, by definition, of all the linear, bilinear, etc., ν-linear symmetric operators belonging to the given algebra G' of the family D^ν.

It is obvious that the family D^1 contains the Lie algebra of the general affine group, and to it there corresponds an affine space. The Lie algebra of the projective group is isomorphic to an algebra of the family D^2 consisting of $T_0 + T_1$ and operators of the 2nd order of the form $f(x)x$, where $f(x)$ is an arbitrary linear form. The Lie algebra of the conformal group with a given scalar product (x, y) is isomorphic to an algebra of the family D^2 consisting of T_0, linear operators of the form $(Ax, y) = -(x, Ay)$ and the identity operator E, and quadratic operators of the form $2(a, x)x - (x, x)a$, where a is an arbitrary vector. The corresponding homogeneous spaces are projective and conformal spaces.

3°. Let $G/H = M$ be a homogeneous space of class R^ν. Fix a definite decomposition $G = H + E$. If in the vector space G one introduces a new law of commutation by putting the commutators in the plane E equal to zero and preserving the values of the commutators of the form $[H, H]$, $[H, E]$, then a new Lie algebra is obtained, which we denote by G'. If g is an element of the algebra G, then the same vector, considered as an element of G', we shall agree to call

denote by g'. The mapping

$$f : g \to g' \tag{7}$$

of G onto G' evidently has the following property: $[f(h), f(g)] = f([h, g])$, where $g \in G,\ h \in H$. According to 1°, the algebra G' is isomorphic to some algebra of a family D^ν acting in E'; hence it follows that the homogeneous space $M' = G'/H'$ is also of class R^ν.

Relying on the mapping (7) introduced above, we shall now construct (locally) a fibered space U with base M and fiber M'. For this purpose we fix a point $x_0 \in M$ and a decomposition $G = H_{x_0} + E_{x_0}$. Further, for each point $x \in M$ we choose g_x such that $g_x x_0 = x$; then, with the aid of the transformation $\operatorname{ad} g_x$, we obtain a completely determined decomposition $G = H_x + E_x$ at the point x. In view of (7), for each point x we have a mapping $f_x : G \to G'_x$, under which $H_x \to H'_x$. If $px = y$ $(p \in G)$, then p induces a mapping of G'_x onto G'_y $(f_y \operatorname{ad} p\, f_x^{-1})$, under which H'_x goes into H'_y; thereby a mapping of M'_x onto M'_y is defined, which we shall denote by the same letter p.

Let now x and $x + dx$ be two infinitely close points, and let g carry x into $x + dx$. We define the transfer of the fiber M'_x onto the fiber M'_{x+dx} as follows:

$$f_x(g) \cdot a \to g \cdot a, \tag{8}$$

where a is the current point of the space M'_x. The constructed mapping does not depend on the choice of g and defines in the fibered space U a connection which, as is not difficult to show, does not depend on the choice of the initial decomposition at the point x_0, nor on the choice of g_x. We shall call this connection **canonical**.

Theorem 2. *The canonical connection is invariant with respect to the group G.*

The proof of the theorem could be derived from Wang' s work ([3]). However, in our case it is easily obtained directly.

Remark. If the group G' is completed to a transitive-differential group $\widetilde{G}$, then, by means of the construction described above, we arrive at a fibered space $\widetilde{U}$ with base M, in which the fiber will be a homogeneous space with group $\widetilde{G}$. The canonical connection constructed in the same way will also be invariant with respect to the group G.

Let us note some particular cases.

Theorem 3. *For there to exist in a homogeneous space of class R^2 an invariant canonical projective connection, it is necessary and sufficient that, for every $h \in H_2$, one have*

$$[[h, x]x] = f(x) \cdot x,$$

where x is an arbitrary element of the plane E, and f is a certain linear form on E.

It can be proved that if the indicated condition is satisfied for some one $h \in H_2$, then it is satisfied also for every $h \in H_2$.

Theorem 4. *For there to exist in a homogeneous space of class R^2 an invariant canonical conformal connection, it is necessary and sufficient that, for every $h \in H_2$, one have*

$$[[h, x]x] = 2(a, x) \cdot x - (x, x) \cdot a,$$

where x is an arbitrary element of the plane E, a depends on h, and (x, y) is a bilinear symmetric form common to all h.

As in the case of the projective connection, it can be proved that if the indicated condition is satisfied for some $h \in H_2$ and, moreover, $(a, a) \neq 0$, then it is satisfied also for every $h \in H_2$.

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CITED LITERATURE

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3. H.-C. Wang, *Nagoya Math. J.*, **13** (1958).

Note: Figure translations are in progress. See original paper for figures.