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Abstract

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ON SMOOTHNESS UP TO THE BOUNDARY OF THE DOMAIN OF THE SPECTRAL FUNCTION OF A SELF-ADJOINT DIFFERENTIAL ELLIPTIC OPERATOR

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The existence of the spectral function, i.e. the kernel of an operator of the type $dE_\lambda/d\rho(\lambda)$, for the case of the Schrödinger equation was established by A. Ya. Povzner ([1]), and in the general case of elliptic equations by Gårding ([2]) and Browder ([3]) (see also ([4])). In this note, for general equations, the behavior of the spectral function near the boundary of the domain is studied. Similar facts have been published only for the Schrödinger equation ([1]). The methods of the note are based on theorems on the local increase of smoothness of solutions of elliptic equations ([5,6]) and on properties of the Green functions of such equations ([7]).

It is convenient for us in § 1° to set forth general questions of expansion in generalized eigenvectors of a self-adjoint operator (for the literature see ([8])), and in § 2° to supplement the theory of expansions for Carleman operators proposed by Mautner ([9]) and developed in one direction or another in a number of works (see ([8])).

1°. Let H_0 be a separable Hilbert space with scalar product $(\cdot,\cdot)_0$; let A be a self-adjoint operator acting in H_0; and let $E(\Delta)$ be its resolution of the identity. An expansion in generalized eigenvectors of A can be constructed as follows. Denote by

$$H_- \supset H_0 \supset H_+$$

some rigging of H_0 by positive and negative spaces, and suppose that the embedding $H_+ \to H_0$ is quasi-nuclear ([10]). There exists a nonnegative finite measure $d\rho(\lambda)$ on the axis $(-\infty,\infty)$ (the spectral density) and an operator function $P(\lambda)$, defined ρ-almost everywhere for all $\lambda \in (-\infty,\infty)$, whose values are Hilbert-Schmidt operators acting from H_+ into H_-, such that

$$E(\Delta) = \int_\Delta P(\lambda)\, d\rho(\lambda). \qquad (1)$$

The range $\mathfrak{R}(P(\lambda))$ of the operator $P(\lambda)$ consists, in a certain sense, of generalized eigenvectors of the operator A corresponding to the number λ. Moreover,

$\|P(\lambda)\| \leq 1$ ($\|\cdot\|$ is the Hilbert-Schmidt norm), $(P(\lambda)u,u)_0 \geq 0$ ($u \in H_+$). Suppose additionally that there exists a linear topological separable space $D \subseteq H_+$, dense in H_+, contained in the domain of definition $\mathfrak{D}(A)$, and mapped continuously by the operator A into H_+. Then every $\varphi \in \mathfrak{R}(P(\lambda))$ satisfies the relation

$$(\varphi, (A-\lambda E)u)_0 = 0 \quad (u \in D).$$

One can choose (in a nonunique way) a system of vectors $\varphi_\alpha(\lambda) \in \mathfrak{R}(P(\lambda)) \subseteq H_-$ ($\alpha = 1, \dots, N_\lambda \leq \infty$) such that

$$(P(\lambda)u,v)_0 = \sum_{\alpha=1}^{N_\lambda} \overline{u_\alpha(\lambda)}\, v_\alpha(\lambda) \quad (u,v \in H_+), \qquad \widetilde{w}(\lambda) = (w_1(\lambda),\, w_2(\lambda), \dots) \tag{2}$$

$$(w_\alpha(\lambda) = (\varphi_\alpha(\lambda), w)_0)$$

is the "Fourier transform" of the vector $w \in H_+$.

The condition that the embedding $H_+ \to H_0$ be quasi-nuclear is also necessary for obtaining (1) for arbitrary A. However, for a fixed A one may impose a less stringent requirement on this embedding: it is sufficient to assume that there exists a continuous bounded function $\gamma(\lambda)$, different from zero on the spectrum of A, such that $\|\gamma(A)J\| < \infty$, where J is an operator mapping H_0 into H_+ ([10]) and considered in H_0. We note that the results of this and the following items are also valid for generalized resolutions of the identity.

2°. Let $H_0 = L_2(Q, dx)$, where Q is a locally compact separable space, and dx is a measure defined on the Borel sets of Q, finite on compact sets and positive on open sets. A self-adjoint operator A in H_0 will be called Carleman if, for $\gamma(\lambda)$ of the form indicated above, the operator $\gamma(A)$ is integral with kernel $K(x,y)$, and

$$\int_Q |K(x,y)|^2\, dx < \infty$$

for almost every $y \in Q$. Choose a function $p(x) \geqslant 1$ ($x \in Q$) so that

$$\int_Q \int_Q |K(x,y)|^2 p^{-1}(y)\, dx\, dy < \infty;$$

then, according to what was said at the end of § 1°, the expansion in generalized eigenfunctions of the operator A can be constructed along the chain

$$H_- \supset H_0 \supset H_+$$

of the form

$$L_2(Q, p^{-1}(x)\, dx) \supset L_2(Q, dx) \supset L_2(Q, p(x)\, dx).$$

Thus, for a Carleman operator every generalized eigenfunction is an ordinary one, but, generally speaking, it does not belong to $L_2(Q, dx)$. For such an operator the following holds.

Theorem 1. The operator $P(\lambda)$ is an integral operator

$$(P(\lambda)u)(x) = \int_Q P(x, y; \lambda)u(y)\,dy \qquad (u \in L_2(Q, p, dx)) \tag{3}$$

with a positive definite kernel $P(x, y; \lambda)$—the spectral function of A—such that

$$\|P(\cdot, \cdot; \lambda)\|_{L_2(Q\times Q, p^{-1}(x)p^{-1}(y)\,dx\,dy)} \leqslant 1.$$

Relation (2) gives the expansion convergent in

$$L_2(Q \times Q, p^{-1}(x)p^{-1}(y)\,dx\,dy):$$

$$P(x, y; \lambda) = \sum_{\alpha=1}^{N_\lambda} \varphi_\alpha(x; \lambda)\varphi_\alpha(y; \lambda) \qquad (\varphi_\alpha(\cdot; \lambda) = \varphi_\alpha(\lambda)). \tag{4}$$

If $P(x, y; \lambda)$ is continuous with respect to $(x, y) \in Q \times Q$, then each of the eigenfunctions $\varphi_\alpha(x; \lambda)$ $(\alpha = 1, \dots, N_\lambda)$ is continuous in $x \in Q$, and the expansion (4) converges absolutely and uniformly on every compact subset of $Q \times Q$.

Let $f(\lambda)$ be bounded on the spectrum of A; then $f(A)$ is also an integral operator (a relation analogous to (3) holds), and its kernel $F(x, y)$ is representable in the form of an absolutely convergent, for almost every (x, y) with respect to the measure $dx\,dy$, integral

$$F(x, y) = \int_{-\infty}^{\infty} f(\lambda)P(x, y; \lambda)\,d\rho(\lambda) \in L_2(Q \times Q, p^{-1}(x)p^{-1}(y)\,dx\,dy). \tag{5}$$

If $P(x, y; \lambda)$ is continuous with respect to $(x, y) \in Q \times Q$ and $\|K(\cdot, y)\|_{L_2(Q,dx)}$ is bounded on every compact subset of Q, then the integral (5), in the case

$$|f(\lambda)| \leqslant C|\gamma(\lambda)|^2,$$

converges absolutely for every x, y and is bounded on every compact subset of $Q \times Q$.

This theorem is proved by developing the arguments in ([4]). We note that if, in addition, the kernel of the operator $(\gamma(A))^*\gamma(A)$ is continuous on $Q \times Q$, then the kernel (5) is also continuous. The growth estimate for the integrals of the eigenfunctions from ([4]) (Theorem 3.3) is also valid for general Carleman operators in $L_2(E_n, dx) = L_2(E_n)$ (E_n is n-dimensional space, dx is Lebesgue measure). Other estimates of the behavior at infinity of eigenfunctions of Carleman operators are given in ([11,12]).

3°. Let $G \subseteq E_n$, generally speaking, be an unbounded domain, and let Γ be its piecewise-smooth boundary. In G consider an elliptic formally self-adjoint differential expression

$$\mathcal{L} = \sum_{|\mu|\leqslant r} a_\mu(x)D^\mu,$$

where

$$a_\mu \in C^{|\mu|}(G \cup \Gamma).$$

On Γ there are given formally self-adjoint boundary conditions

$$(\mathrm{gr}) = (\mathrm{gr})^+,$$

defined by a subspace $W_2^r(\mathrm{gr})$ of the Sobolev space $W_2^r(G)$ (see more details in [13]). The operator in $L_2(G)$

$$u \mapsto \mathcal{L}u$$

$$(u \in W_2^r(\mathrm{gr}) \cap W_{2,0}^r(E_n);\ W_{2,0}^r(G)\text{–finite functions in } W_2^r(G))$$

is, obviously, Hermitian; let A be one of its self-adjoint extensions (in $L_2(G)$ or with values in a broader space), and let $E(\Delta)$ be the corresponding (generalized) resolution of the identity.

Let $a_\mu \in C^{2r+n+p}(G)$ $(p \geq n+1)$; with the aid of Browder' s theorem on the Carleman property of the resolvent kernel of an elliptic operator of high order [3] it is easy to prove that A is a Carleman operator in $L_2(Q, dx) = L_2(G)$, and $\gamma(\lambda) = (\lambda^N - z)^{-1}$ $(N = [\frac{n}{2r}] + 1,\ \operatorname{Im} z \neq 0)$ and $\|K(\cdot, y)\|_{L_2(G)}$ is bounded on every compact subset of G. Thus the results of item 2° are applicable to A. The function $\rho(x)$ may be taken from $C^\infty(G)$; generally speaking, it tends to ∞ as $x \to \Gamma$ and to ∞. Theorem 1, in particular, establishes the existence of a spectral function $P(x, y; \lambda) \in L_2(G \times G, \rho^{-1}(x)\rho^{-1}(y)\, dx\, dy)$. Since in the sense of the theory of generalized functions inside $G \times G$

$$\mathcal{L}_x P(x, y; \lambda) = \lambda P(x, y; \lambda), \qquad \overline{\mathcal{L}}_y P(x, y; \lambda) = \lambda P(x, y; \lambda), \tag{6}$$

it follows, according to [14], that $P(x, y; \lambda)$ will be sufficiently smooth (all derivatives of the form $D_x^\alpha D_y^\beta P(x, y; \lambda)$ exist and are continuous in $G \times G$, $|\alpha|, |\beta| \leq r + n + p$) and (6) are satisfied in the ordinary sense for $x, y \in G$. Every $\varphi \in \mathfrak{R}(P(\lambda))$ also belongs to $C^{r+n+p}(G)$, and the expansion (4) may be differentiated term by term–one may take the indicated derivatives $D_x^\alpha D_y^\beta$. The differentiated expansion (4) will converge absolutely and uniformly on every compact subset of $G \times G$ (the last results are obtained with the aid of [15]).

4°. Let us pass to the formulation of the principal results. In what follows the assumptions of item 3°, ensuring the existence and smoothness of $P(x, y; \lambda)$ inside $G \times G$, are assumed to be fulfilled.

Theorem 2. *Let $r = 2m$ and let the expression $\mathcal{L}$ be properly elliptic in $G \cup \Gamma$ [16]. Suppose that there exists a bounded subdomain $\overline{G}_0 \subseteq G$, adjoining Γ along a piece Γ_0 of class C^{4m+q} $(q \geq n/2)$, such that $a_\mu \in C^{2m+\max(|\mu|, q)}(G_0 \cup \Gamma_0)$. On Γ_0 differential expressions are given*

$$B_j = \sum_{|\mu| \leq m_j} b_{j\mu}(x) D^\mu \quad (b_{j\mu} \in C^{2m+q-1}(\Gamma_0); \quad m_j \leq 2m - 1; \quad j = 1, \dots, m),$$

which are assumed to be normal and to cover $\mathcal{L}$ [16]*; (bc) on* Γ_0 *have the form:* $B_j u|_{\Gamma_0} = 0$ $(j = 1, \dots, m)$*. In addition, suppose that: a) in some* δ*-strip* $\Gamma_{0\delta} \subset \Gamma_0$ *near the boundary* $\dot{\Gamma}_0$ *of the piece of surface* Γ_0*, the expressions* B_j *contain only terms with differentiations along the normal to* Γ_0*; b) in some neighborhood in* G_0 *of the strip* $\Gamma_{0\delta}$*, the expression* $\mathcal{L}$ *contains no terms with mixed derivatives in the normal to* Γ_0 *and tangential directions.*

Then the spectral function $P(x, y; \lambda)$*, corresponding to the ordinary resolution of the identity, belongs to*

$$L_{2,\mathrm{loc}}(G_0 \times G_0,\ ((G_0 \cup \Gamma_0) \times \Gamma_0) \cup (\Gamma_0 \times (G_0 \cup \Gamma_0)))^*.$$

For fixed $y \in G$ $(x \in G)$ *it belongs to* $W^{2m+q}_{2,\mathrm{loc}}(G_0, \Gamma_0)$ *and on* Γ_0 *satisfies the boundary conditions:*

$$B_{j,x} P(x, y; \lambda)\big|_{x \in \Gamma_0} = 0 \quad (\overline{B}_{j,y} P(x, y; \lambda)\big|_{y \in \Gamma_0} = 0); \quad j = 1, \dots, m. \qquad (7)$$

Each eigenfunction $\varphi(x; \lambda) \in \mathfrak{R}(P(\lambda))$ *also belongs to* $W^{2m+q}_{2,\mathrm{loc}}(G_0, \Gamma_0)$ *and satisfies on* Γ_0 *(bc):* $B_{j,x}\varphi(x; \lambda)|_{x \in \Gamma_0} = 0$ $(j = 1, \dots, m)$*. The expansion (4) converges absolutely and uniformly in any bounded domain in* $G \times G$ *which adjoins the boundary* $G \times G$ *inside the piece*

$$((G_0 \cup \Gamma_0) \times \Gamma_0) \cup (\Gamma_0 \times (G_0 \cup \Gamma_0)).$$

Remarks. 1) If G is the interior or exterior of some closed surface Γ and $\Gamma_0 = \Gamma$, then conditions a) and b), naturally, are dropped. They may also be dropped if the (bc) are zero on $\Gamma_{0\delta}$: $B_j|_{\Gamma_{0\delta}} = \partial^{j-1}/\partial\nu^{j-1}$ $(j = 1, \dots, m;$ ν is the normal to $\Gamma_0)$. 2) If the (bc) are zero everywhere on Γ_0 and $\mathcal{L}$ is strongly elliptic in $G_0 \cup \Gamma_0$, then the smoothness restrictions can be weakened: it suffices to assume Γ_0 of class C^{2m+q}, and $a_\mu \in C^{|\mu|+q}(G_0 \cup \Gamma_0)$.

* Let $\mathfrak{D}$ be a bounded domain, γ a piece on its boundary. Recall [5] that we write $u \in W^l_{2,\mathrm{loc}}(\mathfrak{D}, \gamma)$ $(l = 0, 1, \dots$

if $u \in W^l_2(\mathfrak{D}')$ for every domain $\mathfrak{D}'$ having common boundary with $\mathfrak{D}$ only inside the piece γ.

Let us outline the proof. First, it is shown, by means of arguments of the type of (7), that $(A - zE)^{-N}$ $(N = [n/2r] + 1,\ \mathrm{Im}\, z \neq 0)$ is an integral operator with kernel $K(x, y)$ such that the vector-function $K(\cdot, y)$, with values in $L_2(G)$ for $y \in G'_0 \cup \Gamma'_0$, is weakly continuously differentiable up to order $Nr - [n/2] - 1$ inclusive (G'_0 is a subdomain of G_0 adjoining Γ only along the piece Γ'_0 lying inside Γ_0). Then we apply the construction of §2, taking $Q = G \cup \Gamma$, $L_2(Q, dx) = L_2(G)$, $\gamma(\lambda) = (\lambda - z)^{-N}$ (this choice of $\gamma(\lambda)$, different from that in §3, as is easily seen, does not change $P(x, y; \lambda)$). Since $\|K(\cdot, y)\|_{L_2(G)}$ is bounded up to any $\Gamma'_0 \subset \Gamma_0$, the function $\rho(x)$ in §2 may be taken from $C^\infty(G \cup \Gamma)$. This implies the relations: $L_2(G, \rho^{-1}dx) \subset L_{2,\mathrm{loc}}(G_0, \Gamma_0)$ and $L_2(G \times G, \rho^{-1}(x)\rho^{-1}(y)\,dxdy) \subset L_{2,\mathrm{loc}}(G \times G, ((G_0 \cup \Gamma_0) \times \Gamma_0) \cup (\Gamma_0 \times (G_0 \cup \Gamma_0)))$. The smoothness of $P(x, y; \lambda)$ and the relations (7) now follow from the theorem on increasing the smoothness of solutions ([5,6]) and from the fact that (6) are satisfied up to Γ_0 (let us explain that, as the space D of §1, one may now take the suitably topologized set of functions $u \in W_2^r(G)$ which vanish in neighborhoods of $\Gamma \setminus \Gamma_0$ and of ∞, and on Γ_0 satisfy the conditions: $B_j u|_{\Gamma_0} = 0,\ j = 1, \dots, m$).

Theorem 3. *Let G be the interior or the exterior of some closed surface Γ. Suppose that the assumptions of Theorem 2 are fulfilled in the domain $G_0 = G$ (conditions a) and b) are omitted), or the remarks 2 thereto. In that case all derivatives of the form $D_x^\alpha D_y^\beta P(x, y; \lambda)$, where*

$$|\alpha|,\ |\beta| \le r\left\{\min\left(\left[\frac{q}{r}\right], \left[\frac{p-1}{2r}\right]\right) + 1\right\} - \left[\frac{n}{2}\right] - 1,$$

exist and are continuous in $(G \cup \Gamma) \times (G \cup \Gamma)$. The expansion (4) converges absolutely and uniformly in every bounded part of $G \times G$, and it may be differentiated term by term without violating this convergence—namely, one may take precisely the derivatives $D_x^\alpha D_y^\beta$ just indicated.

The proof in the case of bounded G follows from the representation

$$P(x, y; \lambda) = |\lambda - z|^{2N} \int_G \int_G \overline{K(\xi, x)} K(\eta, y) P(\xi, \eta; \lambda)\, d\xi\, d\eta$$

($x, y \in G$) and from the smoothness of $K(\cdot, y)$ indicated above. In the case of unbounded G the proof is considerably more complicated, since the last integral does not exist. It is based on a close representation, the idea of which was suggested by a certain transform of Gårding (see ([2]), §2).

Theorem 3 shows that under its conditions in (7) one may take $y(x)$ on Γ, one may differentiate (7) with respect to $y(x)$, and so on. We note that a similar theorem is also valid for arbitrary G.

In conclusion we observe that, for the study of $P(x, y; \lambda)$, one may also apply a technique using the expression $\mathcal{L}_x + \mathcal{L}_y^+$, developed in ([4,7]).

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CITED LITERATURE

1. A. Ya. Povzner, *Matem. sborn.*, **32** (74), No. 1, 109 (1953).

2. L. Gårding, *12 Congrès des Math. Scand.*, 1953, Lund, 1954, p. 44; *Sborn. per. Matematika*, **1**, No. 3, 1957, p. 107.

3. F. E. Browder, *Proc. Nat. Acad. Sci. U. S. A.*, **40**, No. 6, 454 (1954).

4. Yu. M. Berezanskii, *Matem. sborn.*, **43** (85), No. 1, 75 (1957).

5. Yu. M. Berezanskii, S. G. Krein, Ya. A. Roitberg, *DAN*, **148**, No. 4 (1963).

6. Ya. A. Roitberg, *Ukr. matem. zhurn.*, **15**, No. 4 (1963).

7. Yu. M. Berezanskii, Ya. A. Roitberg, *Ukr. matem. zhurn.*, **15**, No. 2 (1963).

8. Yu. M. Berezanskii, A. G. Kostyuchenko, *Tr. IV Vsesoyuzn. matem. s' ezda*, 1963.

9. F. I. Mautner, *Proc. Nat. Acad. Sci. U. S. A.*, **39**, No. 1, 49 (1953); *UMN*, **10**, No. 4 (1955).

10. Yu. M. Berezanskii, *UMN*, **18**, No. 1, 63 (1963).

11. Yu. M. Berezanskii, *Ukr. matem. zhurn.*, **11**, No. 1, 16 (1959).

12. Yu. M. Berezanskii, Yu. B. Orochko, *Ukr. matem. zhurn.*, **14**, No. 2, 180 (1962).

13. Yu. M. Berezanskii, *DAN*, **131**, No. 3, 478 (1960).

14. Ya. A. Roitberg, *Dokl. AN URSR*, No. 6, 721 (1960).

15. M. G. Krein, *Ukr. matem. zhurn.*, **1**, No. 4, 64 (1949).

16. M. Schechter, *Comm. Pure and Appl. Math.*, **12**, No. 3, 457 (1959); **12**, No. 4, 561 (1959); *Sborn. per. Matematika*, **4**, No. 5, 1960, p. 92; **4**, No. 6, 1960, p. 3.

Note: Figure translations are in progress. See original paper for figures.