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Abstract

Full Text

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APPLICATIONS OF THE DISPERSION METHOD IN BINARY ADDITIVE PROBLEMS

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MATHEMATICS

1. Let $Q(n)$ be the number of solutions of the equation

$$p - \varphi(\xi, \eta) = l, \tag{1}$$

where p runs through the prime numbers, l is a given nonzero integer; $\varphi(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$ is a given binary quadratic form with discriminant not equal to a perfect square; ξ and η independently run through the integers under the condition $0 < \varphi(\xi, \eta) \le n$ (if the discriminant is positive, then the variables ξ and η will also be subject to some inequalities determined by the form φ).

The dispersion method ([1]), in combination with coherent numbers ([2]) and the results of C. Hooley ([3]), makes it possible to give a fairly good lower estimate for $Q(n)$.

Let $k = (a, b, c)$. Then $\varphi(\xi, \eta) = k\varphi_0(\xi, \eta)$, where $\varphi_0(\xi, \eta) = a_0\xi^2 + b_0\xi\eta + c_0\eta^2$ is a primitive form with discriminant $-d = b_0^2 - 4a_0c_0$; $-d = \pm 2^{\beta_0} p_1^{\beta_1} \cdots p_s^{\beta_s}$ is the canonical factorization of the discriminant. Put $P_l = 2^{2\lambda_0} p_1^{2\lambda_1} \cdots p_s^{2\lambda_s}$, where $\lambda_i = 0$ if $p_i \mid l$, and $\lambda_i = 1$ if $p_i \nmid l$ $(i = 0, 1, \dots, s; p_0 = 2)$. Finally, let $r = r_1 r_2 \cdots r_t$, where r_i are the denominators of the rational unimodular substitutions carrying the form $\varphi_0(\xi, \eta)$ into other forms of the genus to which the form $\varphi_0(\xi, \eta)$ belongs; t is the number of classes of forms in the genus; h is the number of classes of primitive forms of discriminant $-d$. We may assume that $a > 0$ and $(r, 2kld) = 1$.

Theorem 1. If $(k, l) = 1$ and $n \to \infty$, then

$$Q(n) \ge C(\varphi) \frac{1}{P_l} \prod_{p/2dl} \frac{(p-1)(p-\chi_d(p))}{p^2 - p + \chi_d(p)} \frac{n}{\ln n} + O\left(\frac{n}{\ln^{1.042} n}\right),$$

where

$$C(\varphi) = \frac{\omega}{hkr^2} L(1, \chi_d) \prod_p \left(1 + \frac{\chi_d(p)}{p(p-1)}\right) \prod_{p/(2d,k)} \left(1 - \frac{1}{p}\right),$$

$\chi_d(m)$ is the Kronecker character; $\omega = 2$ for $d > 4$; $\omega = 4$ for $d = 4$; $\omega = 6$ for $d = 3$; $\omega = 1$ for $d < 0$.

Let us note an interesting consequence of this theorem.

Corollary. There exist infinitely many prime numbers of the form

$$p = \varphi(\xi, \eta) + l. \tag{2}$$

The last assertion, in particular, answers the question of the existence of infinitely many primes (2), posed in the works of W. Sierpiński ([4]) and S. Golomb ([5]) for the form $\varphi(\xi, \eta) = \xi^2 + \eta^2$.

The proof of Theorem 1 is based on the schemes set forth in the monograph of Yu. V. Linnik (([1]), Chapter IX) and in the author’ s note ([6]).

The formulation of the theorem can be simplified if one uses the known formulas for the number of classes h. For example, in the case where $\varphi(\xi, \eta)$ is a positive primitive form of discriminant $-d < 0$, we find that

$$Q(n) \geq \frac{2\pi}{\sqrt{d}} \frac{1}{P_l z^2} \prod_p \left(1 + \frac{\chi_d(p)}{p(p-1)}\right) \prod_{p/2dl} \frac{(p-1)(p-\chi_d(p))}{p^2 - p + \chi_d(p)} \frac{n}{\ln n} + O\left(\frac{n}{\ln^{1.042} n}\right).$$

§ 2. In equation (1) the number l does not depend on the size of n and, consequently, contains a bounded number of small prime divisors. This circumstance substantially facilitates the computations through the introduction of coherent numbers. If, however, the indicated restriction on l is not satisfied, the solution of binary equations related to equation (1) becomes significantly more complicated. Instead of introducing coherent numbers, one has to construct the expected number of solutions of certain equations to which the given binary problem is reduced. Nevertheless, here too, with a suitable choice of the expected number of solutions, the calculations can be simplified. As a result, the dispersion method becomes a very convenient tool for investigating problems of this kind. Let us consider some of them.

I. Let $Q_1(n)$ be the number of solutions of the equation

$$p + xy = n, \tag{3}$$

where p runs over the prime numbers, and x and y independently run over the natural numbers.

Theorem 2. As $n \to \infty$,

$$Q_1(n) = \frac{315\,\xi(3)}{2\pi^4} \prod_{p/n} \frac{(p-1)^2}{p^2 - p + 1}\, n + O\left(\frac{n}{(\ln n)^{1-\varepsilon}}\right), \tag{4}$$

where $\varepsilon > 0$ is an arbitrarily small, but fixed, number.

Since

$$Q_1(n) = \sum_{p<n} \tau(n-p),$$

Theorem 2 is a natural variant of E. Titchmarsh' s problem [7,6]. Formula (4) was previously proved by A. Z. Walfish [8], but only conditionally (on the basis of the extended Riemann hypothesis).

II. Let $Q_2(n)$ be the number of solutions of the equation

$$x_1 x_2 \cdots x_k + xy = n, \tag{5}$$

where $x_1, x_2, \dots, x_k, x$ and y independently run over the natural numbers; $k \geq 2$ is any prescribed natural number.

Theorem 3. As $n \to \infty$,

$$Q_2(n) = \frac{1}{(k-1)!} A_{kB} k(n) n \ln^k n + O(n(\ln n)^{k-1} (\ln\ln n)^{3 \cdot 2^k}),$$

where

$$A_k = \sum_{q=1}^{\infty} \frac{\mu(q)}{q^2} \prod_{p/q} p \left\{ 1 - \left(1 - \frac{1}{p}\right)^{k-1} \right\},$$

$$B_k(n) = \prod_{p/n} \left\{ 1 + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-2} \right\}^{-1} \left\{ 1 + \left(1 - \frac{1}{p}\right)^k \sum_{t=1}^{\alpha(p)} \sum_{m=t}^{\infty} \frac{s_k(m)}{p^m} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-2} \frac{s_k(\alpha(p))}{p^{\alpha(p)}} \right\}.$$

Here

$$s_k(m) = \sum_{\beta_1 + \cdots + \beta_k = m} 1,$$

and $\alpha(p)$ is determined from the canonical factorization

$$n = \prod_{p/n} p^{\alpha(p)}.$$

Theorem 3 contains, as special cases, the results of A. Ingham [9] for $k = 2$, and of E. Titchmarsh [7] and C. Hooley [10] for $k = 3$.

III. Let $Q_3(n)$ be the number of solutions of the generalized Hardy–Littlewood equation

$$p + \varphi(\xi, \eta) = n$$

under the same conditions imposed on the prime p and the binary quadratic form $\varphi(\xi, \eta)$ as in equation (1).

Theorem 4. If $(k, n) = 1$, $n \to \infty$, and $(n, r) = 1$, then

$$Q_3(n) \geq C(\varphi) \frac{1}{P_n} \prod_{p/2dn} \frac{(p-1)(p-\chi_d(p))}{p^2 - p + \chi_d(p)} \frac{n}{\ln n} + O\left(\frac{n}{\ln^{1.042} n}\right).$$

Corollary. Every sufficiently large natural number n, relatively prime to the fixed number r determined by the formula φ, is representable in the form

$$n = p + \varphi(\xi, \eta).$$

3. We give a brief outline of the proof of Theorem 2. Equation (3) is reduced to equations Y_k' of the form

$$x_1' x_2' \cdots x_k' + xy = n, \qquad x_i' \in \Omega_P,$$

where Ω_P is the set of integers all of whose prime divisors exceed

$$P = \exp(\ln n \, \ln\ln\ln n / K \ln\ln n), \quad x < \sqrt{n}\, n_1^{-1}, \quad n_1 = \exp(\ln n)^{\varepsilon}, \quad (x, n) = 1.$$

The equations Y_k' for $k > 6$ are replaced by equations of the form

$$\nu D' + xy = n \tag{6}$$

under the conditions in which $D' \in (D)$ and $\nu \in (\nu)$ run through rectangular regions of certain natural values. Equations (6) are solved by the dispersion method. As the expected number of solutions, for arbitrary D from (D), we take

$$A(n, D) = L_1 \sum_{\substack{x < \sqrt{n}\, n_1^{-1} \\ (x, nD) = 1}} \frac{1}{\varphi(x)},$$

where L_1 is the number of primes $\nu \in (\nu)$. The condition $(x, n) = 1$ does not prevent the application of I. M. Vinogradov' s method and of estimates for Kloosterman sums in solving the main equation of the dispersion method. At the same time, the presence of this condition facilitates the solution of the equations Y_k' for $1 \leq k \leq 6$ at the point where the main lemma of Yu. V. Linnik is applied ((1), Lemma 1, p. 5).

The proof of Theorem 4 is carried out according to the same scheme, with the aid of considerations set forth in the monograph of Yu. V. Linnik ((1), Chapter IX). Naturally, the computations thereby become more complicated.

Theorem 3 is proved by a direct reduction of equation (5) to equations of type (6).

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Note: Figure translations are in progress. See original paper for figures.