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Abstract

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A. L. SHMEL' KIN

FREE POLYNILPOTENT GROUPS

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Denoting by $[A, B]$ the mutual commutator of subgroups A and B in a group G, we shall call, as usual, the lower central series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1}$$

of the group G the series of normal divisors constructed inductively: $G_1 = G$, $G_{i+1} = [G_i, G]$.

Let, further, $n_1, n_2, \ldots, n_i, n_{i+1}, \ldots$ be an arbitrary sequence of integers, $n_i > 1$ $(i = 1, 2, \ldots)$. In an arbitrary group G define a decreasing chain of normal divisors

$$G \supseteq G_{n_1} \supseteq G_{n_1,n_2} \supseteq \cdots \supseteq G_{n_1,n_2,\ldots,n_i} \supseteq G_{n_1,n_2,\ldots,n_i,n_{i+1}} \supseteq \cdots$$

by the rule: G_{n_1} is the above-defined n_1-th term of the lower central series of the group G; $G_{n_1,n_2,\ldots,n_i,n_{i+1}} = (G_{n_1,n_2,\ldots,n_i})_{n_{i+1}}$ is the n_{i+1}-th term of the lower central series of the group $G_{n_1,n_2,\ldots,n_i}$.

If F is a free group with free generators $g_\alpha, \alpha \in M$, then the group

$$G = F/F_{n_1,\ldots,n_k}$$

will be called the free polynilpotent group corresponding to the sequence $n_1, \ldots, n_k$, with free generators $g_\alpha, \alpha \in M$.

For $n_1 = \cdots = n_k = 2$ this definition becomes the definition of a free solvable group of class k; therefore all the results of the present paper are also valid for free solvable groups.

Free polynilpotent groups were studied earlier by K. W. Gruenberg ([1]); he proved that in every such group G

$$\bigcap_{i=1}^{\infty} G_i = E.$$

From the theorem on the uniqueness of the representation of a variety as a product of indecomposable varieties, proved in ([2],[3]), it follows that different sequences $n_1, \dots, n_k$ correspond to distinct free polynilpotent groups.

We shall also introduce analogous notions for Lie rings. In an arbitrary Lie ring L define inductively the ideals L_i: $L_1 = L$, $L_{i+1} = L_i \cdot L$. Just as in the group case, for an arbitrary sequence $n_1, n_2, \dots, n_i, n_{i+1}, \dots$ of integers $n_i > 1$ $(i = 1, 2, \dots)$ define the ideals $L_{n_1, n_2, \dots, n_i}$ $(i = 1, 2, \dots)$ inductively by means of the equality

$$L_{n_1, n_2, \dots, n_i, n_{i+1}} = (L_{n_1, n_2, \dots, n_i})_{n_{i+1}}.$$

If $\mathfrak{C}$ is a free Lie ring with free generators $x_\alpha, \alpha \in M$, then the ring $L = \mathfrak{C}/\mathfrak{C}_{n_1, \dots, n_k}$ will be called the free polynilpotent Lie ring corresponding to the sequence $n_1, \dots, n_k$, with free generators $x_\alpha, \alpha \in M$.

To an arbitrary group G one may associate a Lie ring $\widetilde{G}$ (see ([4])), whose additive group is the direct sum

$$\sum_{i=1}^{\infty} G_i / G_{i+1}$$

of the factors of the lower central series of the group G, and multiplication is defined through the commutation operation in the group G.

It is known that the Lie ring corresponding to a free group is a free Lie ring (see (5)).

Theorem 1. *The Lie ring corresponding to a free polynilpotent group corresponding to the sequence $n_1, \dots, n_k$ is the free polynilpotent Lie ring corresponding to the same sequence.*

In the proof of this theorem one uses the possibility of passing from certain nilpotent Lie rings to a group by means of the Campbell–Hausdorff formula and Lemma 1.

Lemma 1. *The additive group of a free polynilpotent Lie ring is a free abelian group.*

From Lemma 1 and Theorem 1 it follows that

Theorem 2. *The factors of the lower central series of a free polynilpotent group are free abelian groups.**

This theorem has proved useful in the study of free polynilpotent groups. It is used in the proofs of all the theorems below.

The well-known Nielsen–Schreier theorem asserts that every subgroup of a free group is itself free. The problem of describing subgroups can be posed for an arbitrary reduced free group (for the definition see (7)); however, it apparently cannot always have a sufficiently satisfactory solution. In any case, it is clear that only in very rare cases is every subgroup of a reduced free group itself a reduced free group. Therefore the following question naturally arises: which subgroups of a reduced free group are themselves reduced free groups (possibly of another type).

Theorem 3. *Let G be a free polynilpotent group corresponding to the sequence $n_1, \dots, n_k$. Its subgroup U is itself a free polynilpotent group with respect to the same sequence if and only if it has a system of generators $a_\beta,\ \beta \in N$, linearly independent modulo the commutant of the group G.*

A more complete assertion has been obtained for the case of free solvable groups.

Theorem 4. *Let G be a free solvable group of class k. Its subgroup U is a reduced free group if and only if it has a system of generators $a_\beta,\ \beta \in N$, such that for some $i < k$ all a_β lie in the i-th commutant $G^{(i)}$ of the group G and are linearly independent modulo the $(i+1)$-st commutant $G^{(i+1)}$. In this case U is a free solvable group of class $k-i$, and the elements $a_\beta,\ \beta \in N$, are its free generators.*

Subgroups with two generators in a free solvable group also admit a very simple description. Recall (see (8)) that the wreath product of a group A with a group B is constructed as follows. For each $b \in B$ take an isomorphic copy $A(b)$ of the group A. Next take the direct product

$$K = \prod_{b \in B}^{*} A(b)$$

and the splitting extension of the group K by B, in which the elements of B induce the following automorphisms in K: $b_1^{-1} a(b) b_1 = a(bb_1)$, $a \in A,\ b, b_1 \in B$, is called the (discrete) **wreath product** of the group A with the group B and is denoted $A \operatorname{wr} B$.

It is clear that a similar construction can be carried out by taking as the basis not a direct product but, say, a free product, or any verbal (9) and even polyverbal (10) product. By the **l-solvable wreath product** $A \operatorname{wr}_l B$ of the group A with the group B we shall mean a construction similar to the ordinary wreath product, in which the direct product–

* This theorem was proved in (6) for free solvable groups of class 2. In the work of K. Gruenberg (1) it is mentioned that Theorem 2 in its general form is known to P. Hall, but so far I have not encountered the corresponding publication in the journal literature.

is replaced by an l-th soluble product. Here the l-th soluble product of the groups $G_\alpha,\ \alpha \in M$ (see (9)), is the group

$$G = F / \left([G_\alpha]^F \cap F^{(l)} \right),$$

where

$$F = \prod_{\alpha \in M}^{*} G_\alpha,$$

$[G_\alpha]^F$ is the mutual commutant of the subgroups G_α in the group F, and $F^{(l)}$ is the l-th commutant of the group F.

Theorem 5. *Let G be a free soluble group of class k. Any subgroup U of it with two generators is either infinite cyclic, or free soluble of class $l \leq k$, or the l-th soluble wreath product of two infinite cyclic groups, $l < k$; all the possibilities listed are in fact realized.*

Since, obviously, the first soluble product coincides with the direct product, free soluble groups of class 2 can contain only the ordinary wreath product of two infinite cyclic groups.

Theorem 6. *Let L be a free polynilpotent ring of Lie, corresponding to the sequence $n_1, \dots, n_k$ $(k \geq 2)$. If $x, y \in L$ and $xy = 0$, then either x, y are linearly dependent, or $x, y \in L_{n_1,\dots,n_{k-1}}$.*

Hence, and from Theorem 1, it follows:

Theorem 7. *If in a free polynilpotent group G, corresponding to the sequence $n_1, \dots, n_n$ $(k \geq 2)$, the elements g, h commute, then either they lie in one cyclic subgroup, or $g, h \in G_{n_1,\dots,n_{k-1}}$.*

For free soluble groups this theorem was proved by other methods by A. I. Mal' cev (11). Some information on arbitrary subgroups of free polynilpotent groups is given by

Theorem 8. *Every subgroup U of a free polynilpotent group G, corresponding to the sequence $n_1, \dots, n_k$, has an invariant series*

$$E = A_0 \subset A_1 \subset \cdots \subset A_l = U$$

such that A_{j+1}/A_j is the unique maximal noncyclic nilpotent subgroup in U/A_j for all $j \geq 0$, with the possible exception of $j = l - 1$, in which case A_l/A_{l-1} is infinite cyclic.

A simple consequence of Theorem 2 is

Theorem 9. *A free polynilpotent group is linearly orderable.*

A **structural isomorphism of two groups** is an isomorphism between the structures of subgroups of these groups. A group is said to be determined by its structure if it is isomorphic to every group to which it is structurally isomorphic. It is known that free (12) and free nilpotent (13) groups are determined by their structure.

Theorem 10. *Every free polynilpotent group is determined by its structure.*

It follows from Theorem 9 that free polynilpotent groups are R-groups. Therefore for them one may also consider the question of embedding in a complete

R-group, i.e. in such a group G^* in which the equation $x^n = g$ is uniquely soluble for every integer n and every $g \in G^*$. It turns out that such an embedding is always possible. This result is a simple consequence of Theorem 2.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.