Reports of the Academy of Sciences of the USSR

1963

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1963. Volume 152, No. 6

MATHEMATICS

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ON BOUNDARY REGULARITY OF SOLUTIONS OF ELLIPTIC EQUATIONS AND OF CONFORMAL MAPPING

(Presented by Academician V. I. Smirnov on 22 V 1963)

1°. In the first part of this paper we consider the Dirichlet problem with zero boundary condition for the elliptic equation

$$a_{ij}u_{x_ix_j} + b_iu_{x_i} + (c_iu)_{x_i} + du = f_0 + (f_i)_{x_i} \qquad (1 \le i \le n) \tag{1}$$

in a bounded domain $\mathcal{D} \subset R^n$ $(n \ge 2)$ with boundary $\mathfrak{F}\mathcal{D}$. Conditions are given here under which, at a boundary point $\mathfrak{F}\mathcal{D}$, the generalized solution satisfies a Hölder condition or is continuous.

By a solution of the Dirichlet problem with zero boundary condition for equation (1) we shall mean a function $u(x) \in \overset{0}{W}{}_2^{(1)}(D)$ such that

$$\int_D (a_{ij}u_{x_j} + c_iu - f_i)v_{x_i}\,dx = \int_D (b_iu_{x_i} - (a_{ij})_{x_i}u_{x_j} + du - f_0)v\,dx \tag{2}$$

for all $v \in \overset{0}{W}{}_2^{(1)}(\mathcal{D})$.

We assume that for every vector $\vec{\xi} = \{\xi_1, \dots, \xi_n\}$

$$\varkappa_1|\vec{\xi}|^2 \le a_{ij}\xi_i\xi_j \le \varkappa_2|\vec{\xi}|^2, \qquad \text{where } \varkappa_1, \varkappa_2 = \text{const} > 0,$$

and that

$$a_{ij} \in C^{(0,\lambda)}(\overline{\mathcal{D}}) \cap W_{2q}^{(1)}(\mathcal{D}); \qquad b_i, c_i, f_i \in L_{2q}(\mathcal{D});$$

$$d, f_0 \in L_q(\mathcal{D}) \qquad (\lambda > 0,\ 2q > n).$$

Introduce the notation: $\mathfrak{F}\mathcal{E}$ is the boundary, $\mathfrak{C}\mathcal{E}$ the complement of the set $\mathcal{E} \subset R^n$; Σ_ρ is the n-dimensional ball of radius ρ with center at the point $O \in \mathfrak{F}\mathcal{D}$; ω_n is the surface area of the unit sphere; (ρ, ω) are the spherical coordinates of a point $x \in R^n$,

$$\bar{f} = \frac{1}{\omega_n} \int_{\mathfrak{F}\Sigma_\rho} |f| \, d\omega;$$

cap is the Wiener capacity ([1]); k, k_m are positive constants whose value is immaterial to us.

Theorem 1. *Let, at the point $O \in \mathfrak{F}\mathcal{D}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} 2^{(n-2)\nu} \operatorname{cap}(\mathfrak{C}\mathcal{D} \cap \Sigma_{2^{-\nu}}) > 0. \tag{3}$$

Then there exist positive constants α, k such that, for all $\rho > 0$, for the solution of the Dirichlet problem with zero boundary condition the inequality holds

$$\operatorname{ess.}\max_{\Sigma_\rho} |u| \le k\rho^\alpha.$$

In order not to complicate the exposition, we shall prove the theorem for the simplest equation $\Delta u = f$ with $n > 2$. Moreover, suppose that $f(x) \equiv 0$ in some neighborhood $\Sigma_\delta \cap \mathcal{D}$.

Extend the function $u(x)$ to $\complement D$, setting it equal to zero there, and normalize it by the condition $|u| = 1$. From (2) it follows that almost everywhere for $\rho \in [0, \delta]$

$$\int_{\Sigma_\rho} (\nabla u)^2 \, dx = \rho^{n-1} \int_{\mathfrak{F}\Sigma_\rho} u u_\rho \, d\omega. \tag{4}$$

Set

$$U(r, \omega) = \begin{cases} \left(\dfrac{\rho}{r}\right)^{n-2} \left[1 - \left|u\left(\dfrac{\rho^2}{r}, \omega\right)\right|\right], & \text{if } r \ge \rho, \\ 1 - |u(r, \omega)|, & \text{if } r < \rho. \end{cases}$$

For the function $U(r, \omega)$ we have

$$\int_{R^n} (\nabla U)^2 \, dx = 2 \int_{\Sigma_\rho} (\nabla u)^2 \, dx + (n-2)\rho^{n-2} \int_{\mathfrak{F}\Sigma_\rho} (|u| - 1)^2 \, d\omega.$$

Since $U(x) = 1$ for $x \in \Sigma_\rho \cap \complement D$, it follows that

$$\omega_n (n-2) \operatorname{cap}(\Sigma_\rho \cap \complement D) \leq 2 \int_{\Sigma_\rho} (\nabla u)^2 \, dx + (n-2)\rho^{n-2} \int_{\mathfrak{F}\Sigma_\rho} (|u| - 1)^2 \, d\omega. \quad (5)$$

Next note that the first eigenvalue of the V. A. Steklov problem $\Delta u = 0$, $u_n = \lambda u|_{\mathfrak{F}\Sigma_\rho}$ for the ball Σ_ρ is equal to ρ^{-1}. Hence

$$\rho^{n-2} \int_{\mathfrak{F}\Sigma_\rho} u^2 \, d\omega \leq \int_{\Sigma_\rho} (\nabla u)^2 \, dx + \rho^{n-2}\omega_n.$$

From (5) and the last inequality we obtain (cf. ([2],[3]))

$$\frac{n-2}{2(n-1)} \operatorname{cap}(\Sigma_\rho \cap \complement D) \int_{\mathfrak{F}\Sigma_\rho} u^2 \, d\omega \leq \int_{\Sigma_\rho} (\nabla u)^2 \, dx,$$

which, in combination with (4), gives the estimate

$$\int_{\mathfrak{F}\Sigma_\rho} u^2 \, d\omega \leq \int_{\mathfrak{F}\Sigma_R} u^2 \, d\omega \exp\left\{ -\frac{n-2}{n-1} \int_\rho^R \frac{\operatorname{cap}(\Sigma_r \cap \complement D)}{r^{n-1}} \, dr \right\}, \quad (6)$$

valid for almost all $\rho, R \in [0, \delta]$, $\rho \leq R$.

From condition (3) it follows that there exists a constant $\beta > 0$ such that for small ρ

$$\int_\rho^\delta \frac{\operatorname{cap}(\Sigma_r \cap \complement D)}{r^{n-1}} \, dr \geq \beta \ln \frac{\delta}{\rho}.$$

Hence, and from (6), we obtain

$$\rho^{-n} \int_{\Sigma_\rho} u^2 \, dx \leq k_1 \operatorname*{ess\,max}_D u^2 \, \rho^{\frac{n-2}{n-1}\beta}.$$

Now the assertion of the theorem follows from the inequality proved by Moser([4])*

$$\operatorname*{ess\,max}_{\Sigma_{\rho/2}} u^2 \leq k_2 \rho^{-n} \int_{\Sigma_\rho} u^2 \, dx.$$

Similarly one can prove the following

* In ([4]) it is assumed that $\Sigma_\rho \subset D$. However, Moser's proof carries over verbatim to the case considered here as well.

Theorem 2. Suppose that at the point $O \in \mathfrak{F}D$

$$\sum_{\nu=1}^{\infty} 2^{(n-2)\nu} \operatorname{cap}(\mathfrak{C}D \cap \Sigma_{2^{-\nu}}) = \infty. \tag{7}$$

Then the solution of the Dirichlet problem with zero boundary condition is continuous at the point O.

Example 1. Denote by F_ν an arbitrary closed subset of the sphere $\mathfrak{F}\Sigma_{2^{-\nu}}$ with center at the point O ($\nu = 1, 2, ...$), and by α_ν the solid angle under which the set F_ν is seen from the point O. By Theorems 1 and 2, the solution of the Dirichlet problem with zero boundary condition for the domain

$$D = \Sigma_1 \Big/ \bigcup_{\nu=1}^{\infty} F_\nu$$

satisfies the Hölder condition at the point O, if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} \alpha_\nu > 0,$$

and is continuous at O, if

$$\sum_{\nu=1}^{\infty} \alpha_\nu = \infty.$$

Remark 1. Condition (7) coincides with Wiener' s criterion for the regularity of a boundary point for the Laplace equation. From the results of Pucci [(5)], O. A. Oleĭnik [(6)], and Tautz [(7)] it follows that a boundary point is regular for the elliptic equation

$$a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = 0 \tag{8}$$

with sufficiently smooth coefficients if and only if it is regular for the Laplace equation. Quite recently this result was obtained by Hervé [(8)] for equation (8) with coefficients satisfying the Hölder condition in some domain containing $\overline{D}$. On the other hand, Littman, Stampacchia, and Weinberger [(9)] proved the same fact for the uniformly elliptic equation

$$(a_{ij}u_{x_i})_{x_j} = 0$$

with bounded and measurable coefficients.

Theorem 2 formulated above gives a solution of a problem which, although close, nevertheless differs in its formulation from the problem of regular points. It seems to us that the method of proof, which does not use potential theory and barriers, is also of some interest.

2°. Numerous investigations have been devoted to the problem of regularity of a conformal mapping near a boundary point (see, for example, [10,11]). Here we shall present a result adjacent to the questions considered in 1°.

Denote by $\xi(z)$ the function mapping one-to-one and conformally the domain D of the z-plane, bounded by a Jordan curve containing the point $z = 0$, onto the interior of the unit disk of the ξ-plane so that $\xi(0) = 0$.

Let $\Sigma_\rho, \Sigma_\Delta$ $(\rho < \Delta)$ be concentric disks with center $z = 0$ and sufficiently small radii ρ, Δ. Consider the condenser

$$K_{\rho,\Delta} = (\Sigma_\Delta \setminus \Sigma_\rho) \cap D$$

with entrance $\mathfrak{F}\Sigma_\rho \cap D$ and exit $\mathfrak{F}\Sigma_\Delta \cap D$ (for terminology see [12], p. 78).

The resistance $\mathfrak{R}(K_{\rho,\Delta})$ of this conductor is defined as the supremum of the functional*

$$\frac{4\pi}{\displaystyle\int_{K_{\rho,\Delta}} (\nabla f)^2\, dx\, dy}$$

over the set of functions $f \in C(\overline{K}_{\rho,\Delta}) \cap C^{(1)}(K_{\rho,\Delta})$ that are equal to one at the input and to zero at the output of $K_{\rho,\Delta}$.

Theorem 3. *The condition*

$$\varlimsup_{\rho\to 0} \left\{4\alpha \ln \frac{\Delta}{\rho} - \mathfrak{R}(K_{\rho,\Delta})\right\} < +\infty \qquad (\alpha > 0)$$

is necessary and sufficient for the validity of the inequality

$$|\xi(z)| \le k|z|^\alpha. \tag{9}$$

This result is a consequence of the estimate

$$k_1 \exp\left\{-\frac{1}{4}\mathfrak{R}(K_{\rho,\Delta})\right\} \le \max_{|z|=\rho} |\xi(z)| \le k_2(\varepsilon) \exp\left\{-\frac{1}{4}\mathfrak{R}(K_{(1+\varepsilon)\rho,\Delta})\right\},$$

where ε is any sufficiently small positive number.

The simple sufficient condition given below is easily derived from Theorem 3.

Corollary 1. *Let $l(r)$ be the length of the portion of the circle Σ_r contained in $\mathfrak{D}$. If*

$$\overline{\lim_{\rho\to 0}} \left\{ \frac{\alpha}{\pi} \ln \frac{\Delta}{\rho} - \int_{\rho}^{\Delta} \frac{dr}{l(r)} \right\} < +\infty,$$

then inequality (9) is valid.

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Received
15 V 1963

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* In the literature on the theory of functions of a complex variable, in the case of a simply connected domain $K_{\rho,\Delta}$, the term "modulus of a quadrilateral" is used to denote this concept.

Note: Figure translations are in progress. See original paper for figures.