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## Abstract

## Full Text

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# A PRIORI ESTIMATES FOR GENERALIZED SOLUTIONS OF ELLIPTIC AND PARABOLIC EQUATIONS OF SECOND ORDER

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In this paper estimates are established for the maximum of the modulus, the modulus of continuity, and a Liouville-type theorem for solutions of elliptic and parabolic equations of the form

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u + f(x) = 0, \qquad (1)$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u + f(t,x) = \frac{\partial u}{\partial t}; \qquad (2)$$

$$x = (x_1, \dots, x_n); \quad a = \|a_{ij}\|, \quad \lambda^{-1}|\xi|^2 \le (\xi, a\xi), \quad |a_{ij}| \le \lambda. \qquad (3)$$

Regarding the coefficients $b_i, c$, and $f$, it is assumed that they belong to the corresponding spaces $L_p$, where the summability exponents $p$ are greater than or equal to certain “limiting”exponents depending on $n$ (the question of estimates at the limiting exponents is also considered here). The paper develops methods used in ([1–4]). Analogous questions were studied in papers ([1–10]), etc. We note that in the case where the summability exponents $p$ are greater than the “limiting” ones, Theorem 2 was proved in paper ([8]), while a result close to Theorem 4B, under stronger a priori assumptions on the solution $u(t,x)$, was established in paper ([10]).

**1.** We shall denote by $\Omega$ a domain in $E_n(x_1, \dots, x_n)$, by $K_r$ the ball $\{|x| \le r\}$, and by $Q$ and $Q_r^h$ respectively the cylinders $\{\Omega \times [T_1, T_2]\}$ and $\{K_r \times [-h, 0]\}$.

A **generalized solution** of equation (1) (or (2)) in $\Omega$ (in $Q$) is a function $u(x) \in W_2^1(\Omega)$ $(u(t,x) \in W_2^1(Q))$ satisfying the following identity for every $\varphi(x) \in \overset{\circ}{C}{}^\infty(\Omega)$ $(\varphi(t,x) \in C^\infty(Q),\ \varphi(t,x) \in \overset{\circ}{C}{}^\infty(\Omega)$ for $T_1 \le t \le T_2)$:

$$\int_\Omega (\varphi_x, au_x)\,dx = \int_\Omega \varphi\big[(b,u_x) + cu + f\big]\,dx \qquad (4)$$

or

$$\iint_Q [\varphi u_t + (\varphi_x, a u_x)]\, dx\, dt = \iint_Q \varphi[(b, u_x) + cu + f]\, dx\, dt, \tag{5}$$

where $u_x = \operatorname{grad}_x u,\ \varphi_x = \operatorname{grad}_x \varphi,\ u_t = \partial u/\partial t,\ b = (b_1, \dots, b_n)$.

If the function $u(x)$ (or $u(t,x)$) satisfies (4) (or (5)), where instead of the sign $=$ there stands the sign $\le$ and $\varphi(x) \ge 0$ (or $\varphi(t,x) \ge 0$), then $u(x)$ (or $u(t,x)$) is called a **generalized subsolution** in $\Omega$ (or in $Q$).

In the proofs the following known inequality is used:

**Lemma 1.** Let $u(x) \in W_2^1$ in $K_\rho$, $N \subseteq K_\rho$, and $\operatorname{mes} N \ge c_0 \operatorname{mes} K_\rho$. Then

$$\left(\rho^{-n} \int_K u^{2k}\, dx\right)^{1/k} \le c \left[\rho^{2-n} \int_{K_\rho} u_x^2\, dx + \left(\rho^{-n} \int_N u^{2q}\, dx\right)^{1/q}\right],$$

where $1 \le k \le n/(n-2)$, $q \ge 1$, the constant $c$ depends on $n$ and $c_0$; if $u(x) \in \overset{\circ}{W}{}_2^1$ in $K_\rho$, then the second term on the right-hand side may be omitted (for $n = 2$ the number $k$ may be arbitrary $\ge 1$, but the constant $c$ also depends on $k$).

In what follows, by $c$ we denote constants depending on $n$ and $\lambda$, and also on the numbers $p$ and $s$, which will be specified below.

## 2. Equations of elliptic type.

**Theorem 1.** Let $v(x)$ be a nonnegative subsolution of equation (1) in the ball $K_{3r}$; let

$$\|c(x)\|_{L_p(K_{3r})} + \|f(x)\|_{L_p(K_{3r})} = c_1, \qquad p > n/2,$$

$$b_i \in L_s(K_{3r}), \qquad s = n \text{ for } n \ge 3, \quad s > 2 \text{ for } n = 2,$$

and let

$$c c_1 r^{2\gamma} \le 1$$

and

$$c r^{1-n/s} \|b_i(x)\|_{L_s(K_{3r})} \le 1,$$

where $\gamma = 1 - n/2p$. Then

$$\operatorname*{vrai\,max}_{K_r} v^2 \le c \left(r^{-n} \int_{K_{3r}} v^2\, dx + 1\right).$$

If $p = n/2$ for $n \ge 3$, then $v(x)$ is summable with any power in $K_r$ (for $n = 2$ this follows directly from the embedding theorem (see Lemma 1)).

Let $w = v^m$, $m \geq 1$; substituting the corresponding test functions into the integral inequality for $v(x)$ and applying Lemma 1, we obtain the following estimate for $k > q = p/(p-1)$:

$$\left(\rho^{-n}\int_{K_\rho} w^{2k}\,dx\right)^{1/k} \leq c\left[\left(m^3 + \frac{\rho^2}{\sigma^2}\right)\left((\rho+\sigma)^{-n}\int_{K_{\rho+\sigma}} w^{2q}\,dx\right)^{1/q} + m^{-2m}\right].$$

On the basis of this inequality one carries out an iteration process, which also completes the proof of Theorem 1.

**Theorem 2.** If $u(x)$ is a generalized solution of equation (1) in $K_{3r}$ and the hypotheses of Theorem 1 on its coefficients are satisfied ($p > n/2$), then for $\rho \leq r$

$$\operatorname*{osc}_{K_\rho} u \leq c_2\rho^\alpha,$$

where $\alpha > 0$ depends only on $n, \lambda, \gamma$, while $c_2$ depends on $n, \lambda, \gamma, r, \|u\|_{L_2(K_{3r})}$.

By virtue of Theorem 1, for $v = |u|$, which is a subsolution for an equation of the form (1), the function $u(x)$ is bounded in $K_r$ (hence one may assume that $c(x) = 0$ in equation (1)). Let

$$M = \operatorname*{vrai\,max}_{K_{4\rho}} \pm u, \qquad 4\rho \leq r_0(n,\lambda,\gamma) < r;$$

denote by $w(x)$ that one of the functions

$$\ln^+[1 \pm (u/M) + \varepsilon]$$

which is equal to 0 on the set $N \subset K_{3\rho}$ and

$$\operatorname{mes} N \geq \tfrac{1}{2}\operatorname{mes} K_{3\rho}$$

(the number $\varepsilon > 0$ depends only on $n$ and $\lambda$). Applying Theorem 1 to $w$, we establish the dilemma: either

$$\operatorname*{osc}_{K_\rho} u \leq 2\rho^\gamma,$$

or

$$\operatorname*{osc}_{K_\rho} u \leq \eta \operatorname*{osc}_{K_{4\rho}} u,$$

where $0 < \eta < 1$, from which Theorem 2 follows easily.

**Remark.** Theorems 1 and 2 are also valid for more general equations:

$$Lu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x),$$

where $f_i \in L_{2p}$, $p > n/2$. Let us also note that in the proof of Theorems 1 and 2, for $p > (n+1)/2$ one may apply the device consisting in the introduction of new variables (see [4], item 6).

## 3. Equations of parabolic type.

**Theorems 3A and 3B.** Let $v(t,x) \geq 0$ be a nonnegative generalized subsolution of equation (2) in $Q_{2r}^{4r^2}$, and let one of the following conditions A or B be satisfied:

**A.** The functions

$$b_i(t,x) \in L_s(Q_{2r}^{4r^2}), \qquad s = n+2 \text{ for } n \geq 3, \quad s > 4 \text{ for } n = 2,$$

$$\|c(t,x)\|_{L_p(Q_{2r}^{4r^2})} + \|f(t,x)\|_{L_p(Q_{2r}^{4r^2})} = c_3 < \infty, \qquad p > \frac{n}{2} + 1,$$

and

$$cr^{1-(n+2)/s}\|b_i(t,x)\|_{L_s(Q_{2r}^{4r^2})} \leq 1$$

and

$$cc_3 r^{2\gamma} \leq 1,$$

where

$$\gamma = 1 - \frac{n+2}{2p}.$$

**B.** For $-4r^2 \leq t \leq 0$, the functions

$$b_i(t,x) \in L_s(K_{2r}), \qquad s = n \text{ for } n \geq 3, \quad s > 2 \text{ for } n = 2,$$

$$\left(\|c(t,x)\|_{L_p(K_{2r})} + \|f(t,x)\|_{L_p(K_{2r})}\right) \leq c_4 < \infty, \qquad p > n/2,$$

and

$$cr^{1-n/s}\|b_i(t,x)\|_{L_p(K_{2r})} \leq 1, \qquad cc_4 r^{2\gamma} \leq 1,$$

where

$$\gamma = 1 - \frac{n}{2p}.$$

Then

$$\operatorname*{vrai\,max}_{Q_r^{r^2}} v^2(t,x) \leq c\left[r^{-n-2}\iint_{Q_{2r}^{4r^2}} v^2\,dx\,dt + 1\right].$$

If, for $n \geq 3$, in condition A or B respectively $p = (n+2)/2$, or $p = n/2$, then $v(t,x)$ is summable in $Q_r^{r^2}$ with any power.

Under the condition that $b_i(t,x) = c(t,x) = f(t,x) = 0$, this theorem was established in paper ([3]).

**Theorems 4A and 4B.** Let $u(t,x)$ be a generalized solution of equation (2) in $Q_{2r}^{4r^2}$. Then, under assumption A or B on the coefficients of equation (2) (see Theorems 3A and 3B), for $p \leq r$

$$\operatorname*{osc}_{Q_\rho^{\rho^2}} u \leq c_5\rho^\alpha,$$

where $\alpha > 0$ depends only on $n, \lambda, \gamma$, and $c_5$ on $n, \lambda, \gamma, r, \|u\|_{L_2(Q_{2r}^{4r^2})}$.

We shall consider here the case where in equation (2) $c = f = 0$; in this case the following Liouville-type theorem will also be established:

**Theorem 5.** Let $u(t,x)$ be a generalized solution of equation (2) (with $c = f = 0$) in the half-space $\{t \le 0\}$; let

$$\lambda^{-1}|\xi|^2 \le (\xi, a, \xi), \qquad |a_{ij}| \le \mu$$

and, for $n \ge 3$,

$$\text{a)} \quad (\lambda + \lambda^2) \left( \iint_{\{t \le 0\}} |b_i|^{n+2} \, dx \, dt \right)^{2/(n+2)} \le c(n)$$

or

$$\text{b)} \quad \operatorname*{vrai\,max}_{-\infty < t \le 0} \lambda^2 \left( \int_{E_n} |b_i(t,x)|^n \, dx \right)^{2/n} \le c(n),$$

and for $n = 2$, $b_i = 0$.* Then there exists an $\alpha_0$, depending only on $n, \lambda, \mu$, such that from the condition

$$|u(t,x)| \le H(r^\alpha + 1) \quad \text{for } (t,x) \in Q_r^{r^2}, \qquad 0 < \alpha < \alpha_0$$

($H$ is a certain constant) it follows that $u = \text{const}$.

Let $v(t,x)$ be a nonnegative subsolution in $Q_{2r}^{4r^2}$. By means of the iterative process (see ([3])) we establish the estimate **

$$\operatorname*{vrai\,max}_{Q_r^{r^2}} v^2(t,x) \le c r^{-n-2} \iint_{Q_{2r}^{4r^2}} v^2 \, dx \, dt. \tag{6}$$

**Lemma 2.** Let $u(t,x) \ge 0$ be a generalized solution in $Q_r^{r^2}$ and

$$\operatorname{mes}\{(t,x) \in Q_r^{r^2};\ u(t,x) \ge 1\} \ge \frac{1}{2} \operatorname{mes} Q_r^{r^2}.$$

Then there exist numbers $\theta$ and $c$, $0 < \theta < 1$, depending only on $n, \lambda, \mu$, such that

$$u(t,x) \ge c > 0 \quad \text{in } Q_{\theta r}^{\theta^2 r^2}.$$

We first prove the following assertion: there exist numbers $\alpha, \beta$, and $h$, depending only on $n, \lambda, \mu$ $(0 < \alpha, \beta, h < 1)$, such that for $-\alpha r^2 \le t \le 0$

$$\operatorname{mes} N_t\{x \in K_{\beta r};\ u(t,x) \ge h\} \ge \frac{1}{4} \operatorname{mes} K_{\beta r}.$$

Let

$$\mu(t) = \operatorname{mes}\{x \in K_r;\ u(t,x) \ge 1\}$$

and $0 < \alpha < \frac{1}{2}$; it is obvious that there exists

$$\tau \in [-r^2, -\alpha r^2]$$

such that

$$\mu(\tau) \ge (1/2 - \alpha)(1-\alpha)^{-1} \operatorname{mes} K_r.$$

Let

$$v = f(u) = \ln^+ \frac{1}{u+h}, \qquad 0 < h < \frac{1}{2};$$

put, in the integral identity for $u(t,x)$,

$$\varphi = f'(u)\eta^2(y)\xi(t),$$

where $y = |x|$, $\eta(y) = 1$ for $0 \le y \le \beta r < r$, $\eta(y) = 0$ for $y \ge r$, and $\eta(y)$ is linear for $\beta r \le y \le r$; $\xi(t) = 1$ for $\tau \le t \le s \in [-\alpha r^2, 0]$, $\xi(t) = 0$ for $t < \tau$ and $t > s$. We obtain the estimate

$$\int_{K_{\beta r}} v(s,x)\,dx + \frac{1}{4\lambda}\int_\tau^s \int_{K_{\beta r}} v_x^2\,dx\,dt \le c\beta^{-n}(1-\beta)^{-1} \operatorname{mes} K_{\beta r} + \int_{K_r} v(\tau,x)\,dx;$$

hence

$$\ln\frac{1}{2h} \operatorname{mes}[K_{\beta r} \setminus N_s] \le c\beta^{-n}(1-\beta)^{-1} \operatorname{mes} K_{\beta r} + \ln\frac{1}{h}\big(\operatorname{mes} K_r - \mu(\tau)\big) \le$$

$$\le \left[c\beta^{-n}(1-\beta)^{-1} + \ln\frac{1}{h}\, 2^{-1}\beta^{-n}(1-\alpha)^{-1}\right] \operatorname{mes} K_{\beta r}.$$

We choose first $\alpha$ and $\beta$ so that

$$\frac{1}{2}(1-\alpha)\beta^n \leq \frac{2}{3},$$

and then $h$ so small that

$$\operatorname{mes}[K_{\beta r} \setminus N_s] \leq \frac{3}{4} \operatorname{mes} K_{\beta r}$$

for $-\alpha r^2 \leq s \leq 0$, whence the assertion being proved follows.

* Other conditions, analogous to those under which Theorem 3 is proved in paper ([4]), may also be imposed on the coefficients $b_i$ in this theorem.

** Here and below the constants $c$ depend on $n, \lambda$, and $\mu$.

Consider now $w = g(u) = \ln^+ \dfrac{h}{u+\varepsilon}$, where the positive number $\varepsilon \ll h/2$ will be chosen below; just as above, we obtain the estimate

$$\int_{-\alpha r^2}^{0} \int_{K_{3r}} w_x^2 \, dx \, dt \leq c \operatorname{mes} K_{3r} + c \int_{K_r} w(-\alpha r^2, x) \, dx \leq c r^n \ln \frac{3h}{\varepsilon}. \qquad (7)$$

Applying Lemma 1 to $w(t,x)$ for $-\alpha r^2 \leq t \leq 0$ $(k=1)$, noting that $w(t,x) = 0$ for $x \in N_t$, $\operatorname{mes} N_t \geq \frac{1}{4} \operatorname{mes} K_{3r}$, and taking (7) into account, we find that

$$\iint_{Q_{3r}^{\alpha r^2}} w^2 \, dx \, dt \leq c r^{n+2} \ln \frac{3h}{\varepsilon}.$$

Since $w(t,x)$ is a nonnegative subsolution, by estimate (6), for some $\theta$, $0 < \theta = \theta(n, \lambda, \mu) < 1$, for $(t,x) \in Q_{\theta r}^{\theta^2 r^2}$,

$$w^2(t,x) \leq c \ln \frac{3h}{\varepsilon}.$$

At any point $(t,x) \in Q_{\theta r}^{\theta^2 r^2}$ either $u(t,x) \geq h - \varepsilon \geq h/2$, or $u < h - \varepsilon$; in the second case, for sufficiently small $\varepsilon = \varepsilon(n, \lambda, \mu)$,

$$\ln^2 \frac{h}{u+\varepsilon} \leq c \ln \frac{3h}{\varepsilon} < \left( \ln \frac{h}{\sqrt{\varepsilon}} \right)^2,$$

whence $u \geq \sqrt{\varepsilon} - \varepsilon = \sqrt{\varepsilon}(1 - \sqrt{\varepsilon})$.

For the proof of Theorems 4B and 5, note that the generalized solution $u(t,x)$ is bounded in $Q_r^{r^2}$, and put $M = \operatorname{vrai\,max} \pm u(t,x)$ in $Q_r^{r^2}$; applying Lemma 2 to that one of the functions $1 \pm u/M$ which is $\geq 1$ on a set of measure $\geq \frac{1}{2} \operatorname{mes} Q_r^{r^2}$, we find that

$$\operatorname*{osc}_{Q_{\theta r}^{\theta^2 r^2}} u \leq \eta \operatorname*{osc}_{Q_r^{r^2}} u, \qquad \eta = \eta(n, \lambda, \mu) < 1,$$

from which Theorems 4B and 5 follow.

4. The following examples show that in Theorems 1, 2, 3B, and 4B one cannot decrease the exponents $p$ and $s$ of summability of the coefficients $b_i$, $c$, and $f$.

**Example 1.** $u = (-\ln r)^\alpha$, $r = |x|$, $0 < \alpha < 1/3$,

$$\Delta u = \alpha r^{-2}(-\ln r)^{\alpha-2}[\alpha - 1 + (n-2)\ln r] = f = \alpha(r \ln r)^{-2}[\alpha - 1 + (n-2)\ln r]u = cu$$

$$= \sum_{i=1}^{n} [\alpha - 1 + (n-2)\ln r]\, r^{-2}(\ln r)^{-1} x_i \frac{\partial u}{\partial x_i} = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i};$$

here $p = n/2$, $s < n$ for $n \geq 3$, $s = 2$ for $n = 2$; the solution is unbounded.

**Example 2.** $u = r^\alpha$, $0 < -\alpha < n/2 - 1$ $(n \geq 3)$,

$$\Delta u = \alpha r^{\alpha-2}(\alpha - 2 + n) = f = cu = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i};$$

here $p < n/2$, $s < n$; the solution is summable only to a certain power.

The following example shows that Theorems 2 and 4B for $p = n/2$, $s < n$ $(n \geq 3)$, $s = 2$ $(n = 2)$ do not hold even under the additional assumption that the solution is bounded.

**Example 3.** $u = 1/\ln r$,

$$\Delta u = r^{-2} \ln^{-3} r\, [2 + (2-n)\ln r] = f = cu = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$$

(see Example 1).

Analogous examples can also be constructed for Theorems 3A and 4A.

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*Note: Figure translations are in progress. See original paper for figures.*