AN ASYMPTOTIC METHOD FOR INVESTIGATING TRANSIENT PROCESSES IN NONLINEAR OSCILLATORY SYSTEMS

We consider a system of nonlinear differential equations

1963

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

T. A. TIBILOV

AN ASYMPTOTIC METHOD FOR INVESTIGATING TRANSIENT PROCESSES IN NONLINEAR OSCILLATORY SYSTEMS

(Presented by Academician N. N. Bogolyubov on 28 V 1963)

We consider a system of nonlinear differential equations

$$\sum_{k=1}^{n}(a_{jk}D^2+b_{jk}D+c_{jk})\,x_k=\mu Q_j(t,\theta_1,\dots,\theta_m,x_1,\dots,x_n,Dx_1,\dots,Dx_n)$$

$$(j=1,\dots,n), \tag{1}$$

where D is the differential operator d/dt; x_k are unknown functions of time t; a_{jk},b_{jk},c_{jk} are given constants; $Q_j(t,\theta_1,\dots,\theta_m,x_1,\dots,x_n,Dx_1,\dots,Dx_n)$ are known functions of their arguments and, moreover, periodic functions of $\theta_1,\dots,\theta_m$ with period 2π, with $D\theta_q=\alpha_q$ $(q=1,\dots,m)$; μ is a small parameter.

We seek a manifold of particular solutions of the system of differential equations (1) describing a transient process in a nonlinear oscillatory system with many degrees of freedom. A mathematical justification of the method for solving system (1) set forth here is given in ([1]).

Let the characteristic equation

$$\Delta(D)=\left|a_{jk}D^2+b_{jk}D+c_{jk}\right|=0 \tag{2}$$

have s' real roots χ_σ $(\sigma=1,\dots,s')$ and s'' pairs of complex-conjugate roots $\varepsilon_h\pm i\omega_h$ $(h=1,\dots,s'')$.

We introduce the following assumptions: a) the determinant $\Delta(D)$ has only simple roots; b) the determinant of the coefficients of the highest derivatives in (1) is nonzero; c) all fractions $F_{kj}(D)/\Delta(D)$, where $F_{kj}(D)$ is the algebraic complement of the element $a_{jk}D^2+b_{jk}D+c_{jk}$ in $\Delta(D)$, are proper.

Under these assumptions it is possible to transform system (1) to normal coordinates ([2]).

The transformation formulas have the form

$$x_j = \sum_{\sigma=1}^{s'} v_{j\sigma}\xi_\sigma + \sum_{h=1}^{s''} N_{jh} a_h \cos(u_h + \gamma_{jh}),$$

$$Dx_j = \sum_{\sigma=1}^{s'} v_{j\sigma}\chi_\sigma\xi_\sigma + \sum_{h=1}^{s''} N_{jh} a_h \{\varepsilon_h \cos(u_h + \gamma_{jh}) - \omega_h \sin(u_h + \gamma_{jh})\}$$

$$(j = 1, \dots, n). \tag{3}$$

Here ξ_σ, a_h, u_h are new variables (normal coordinates), satisfying the equations

$$\frac{d\xi_\sigma}{dt} = \chi_\sigma\xi_\sigma + \frac{\mu}{\Delta'(\chi_\sigma)} \sum_{k=1}^{n} w_{\sigma k} Q_k(t, x_1, \dots, x_n, Dx_1, \dots, Dx_n, \theta_1, \dots, \theta_q),$$

$$\frac{da_h}{dt} = \varepsilon_h a_h + 2\mu \,\mathrm{Re}\left[\frac{e^{-iu_h}}{\Delta'(\varepsilon_h + i\omega_h)} \sum_{k=1}^{n} W_{s'+h,k} Q_k(t, x_1, \dots, x_n, Dx_1, \dots, Dx_n, \theta_1, \dots, \theta_q)\right], \tag{4}$$

$$\frac{du_h}{dt} = \omega_h + \frac{2}{a_h}\mu \times \tag{4}$$

$$\times \,\mathrm{Im}\left[\frac{e^{-iu_h}}{\Delta'(\varepsilon_h + i\omega_h)} \sum_{k=1}^{n} W_{s'+h,k} Q_k(t, x_1, \dots, x_n, Dx_1, \dots, Dx_n, \theta_1, \dots, \theta_q)\right]$$

$$(\sigma = 1, \dots, s';\ h = 1, \dots, s'').$$

The procedure for finding the quantities $v_{j\sigma}, N_{jh}, \gamma_{jh}, \mathfrak{w}_{\sigma k}, W_{s'+h,k}$, entering formulas (3) and equations (4), is given in the paper ([2]). The variables $x_1, \dots, x_n, Dx_1, \dots, Dx_n$ in equations (4) are assumed to have been replaced by their expressions (3).

With the aid of the relations

$$\xi_\sigma = \xi_{0\sigma} e^{\varkappa_\sigma t}, \qquad a_h = a_{0h} e^{\varepsilon_h t} \tag{5}$$

we introduce the new variables $\xi_{0\sigma}$ and a_{0h}. Passing in equations (4) to the variables $\xi_{0\sigma}, a_{0h}$, we obtain

$$\frac{d\xi_{0\sigma}}{dt} = \mu \Phi_\sigma(\xi_0, a_0, u, \theta),$$

$$\frac{da_{0h}}{dt} = \mu \Phi^{(1)}_{s'+h}(\xi_0, a_0, u, \theta), \qquad (6)$$

$$\frac{du_h}{dt} = \omega_h + \mu \Phi^{(2)}_{s'+h}(\xi_0, a_0, u, 0).$$

In equations (6) the following notation is used:

$$\Phi_\sigma(\xi_0, a_0, u, \theta) =$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{e^{-\varkappa_\sigma t}}{\Delta'(\varkappa_\sigma)} \sum_{k=1}^{n} \mathfrak{w}_{\sigma k} Q_k(t, x_1, \dots, x_n, Dx_1, \dots, Dx_n, \theta_1, \dots, \theta_q)\, dt, \quad (7)$$

$$\Phi^{(1)}_{s'+h}(\xi_0, a_0, u, \theta) = 2\,\mathrm{Re}\,\Phi_{s'+h}, \qquad \Phi^{(2)}_{s'+h}(\xi_0, a_0, u, \theta) = \frac{2}{a_h}\,\mathrm{Im}\,\Phi_{s'+h},$$

where

$$\Phi_{s'+h} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{e^{-\varepsilon_h t - iu_h}}{\Delta'(\varepsilon_h + i\omega_h)} \sum_{k=1}^{n} W_{s'+h,k} \times \qquad (8)$$

$$\times Q_k(t, x_1, \dots, x_n, Dx_1, \dots, Dx_n, \theta_1, \dots, \theta_q)\, dt,$$

and by $\xi_0, a_0, u, \omega, \theta$ is denoted the collection of quantities $\xi_{01}, \dots, \xi_{0s'};\ a_{01}, \dots, a_{0s''};\ u_1, \dots, u_{s''};\ \omega_1, \dots, \omega_{s''};\ \theta_1, ..$ Here in the expressions (7), (8) the integration is performed with respect to the explicitly contained time t.

We proceed to finding approximate solutions of the system of equations (6) in the resonance case, when the frequencies $\omega_1, \dots, \omega_{s''}, \alpha_1, \dots, \alpha_q$ satisfy the condition

$$k_1\omega_1 + \dots + k_{s''}\omega_{s''} + l_1\alpha_1 + \dots + l_q\alpha_q = 0, \qquad (9)$$

where $k_1, \dots, k_{s''}, l_1, \dots, l_q$ are integers.

Let the functions $\Phi_\sigma(\xi_0, a_0, u, \theta)$, $\Phi^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta)$ $(\nu = 1, 2)$ be sums of the form

$$\Phi_\sigma(\xi_0, a_0, u, \theta) = \sum{}' \Phi_{\sigma,k,l}(\xi_0, a_0) e^{i(ku + l\theta)}, \qquad (10)$$

$$\Phi^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta) = \sum{}' \Phi^{(\nu)}_{s'+h,k,l}(\xi_0, a_0) e^{i(ku+l\theta)}$$

$$(\sigma = 1, \dots, s';\ h = 1, \dots, s'';\ \nu = 1, 2),$$

where

$$\Phi_{\sigma,k,l}(\xi_0, a_0) = \frac{1}{(2\pi)^{s''+q}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Phi_\sigma(\xi_0, a_0, u, \theta) e^{-i(ku+l\theta)}\, du\, d\theta; \qquad (11)$$

$$\Phi^{(\nu)}_{s'+h}(\xi_0, a_0) = \frac{1}{(2\pi)^{s''+q}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Phi^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta) e^{-i(ku+l\theta)}\, du\, d\theta. \qquad (11)$$

In expressions (10), (11), for brevity of notation the letters k, l denote, respectively, the collections of quantities $k_1, \dots, k_{s''}, l_1, \dots, l_q$, and the prime on the summation sign indicates that the summation extends over those values $k_1, \dots, k_{s''}, l_1, \dots, l_q$ which satisfy condition (9).

Let us introduce into consideration the functions

$$\begin{aligned} F_\sigma(\xi_0, a_0, u, \theta) &= \sum \frac{\exp i(ku + l\theta)}{i(k\omega + l\alpha)} \Phi_{\sigma,k,l}(\xi_0, a_0), \\ F^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta) &= \sum \frac{\exp i(ku + l\theta)}{i(k\omega + l\alpha)} \Phi^{(\nu)}_{s'+h,k,l}(\xi_0, a_0). \end{aligned} \qquad (12)$$

In formulas (12) the summation extends over those numbers $k_1, \dots, k_{s''}, l_1, \dots, l_q$ which do not satisfy condition (9). For the functions $F_\sigma(\xi_0, a_0, u, \theta)$, $F^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta)$ the identities

$$\begin{aligned} \omega \frac{\partial F_\sigma}{\partial u} + \alpha \frac{\partial F_\sigma}{\partial \theta} &= \Phi_\sigma(\xi_0, a_0, u, \theta) - \sum{}' \Phi_{\sigma,k,l}(\xi_0, a_0) e^{i(ku+l\theta)}, \\ \omega \frac{\partial F^{(\nu)}_{s'+h}}{\partial u} + \alpha \frac{\partial F^{(\nu)}_{s'+h}}{\partial \theta} &= \Phi^{(\nu)}_{s'+h}(\xi_0, a_0, u, \theta) - \sum{}' \Phi^{(\nu)}_{s'+h,k,l}(\xi_0, a_0) e^{i(ku+l\theta)} \end{aligned} \qquad (13)$$

hold.

With the aid of the relations

$$\begin{aligned} \xi_{0\sigma} &= \xi_{10\sigma} + \mu F_\sigma(\xi_{10}, a_{10}, u_{10}, \theta), \\ a_{0\sigma} &= a_{10\sigma} + \mu F^{(1)}_{s'+h}(\xi_{10}, a_{10}, u_{10}, \theta), \\ u_h &= u_{10h} + \mu F^{(2)}_{s'+h}(\xi_{10}, a_{10}, u_{10}, \theta), \end{aligned} \qquad (14)$$

where $F_\sigma(\xi_{10}, a_{10}, u_{10}, \theta)$, $F^{(\nu)}_{s'+h}(\xi_{10}, a_{10}, u_{10}, \theta)$ are periodic functions with period 2π in the variables $u_{101}, \dots, u_{10s''}, \theta_1, \dots, \theta_q$, we introduce certain functions of time $\xi_{101}, \dots, \xi_{10s'}, a_{101}, \dots, a_{10s''}, u_{101}, \dots, u_{10s''}$, which must be determined from a system of differential equations. Substituting (14) into (6) and taking into account condition (9) and identities (13), we obtain

$$\begin{aligned}
\frac{d\xi_{10\sigma}}{dt} &= \mu \sum{}' \Phi_{\sigma, k_1, \dots, k_{s''}, l_1, \dots, l_q}(\xi_{101}, \dots, \xi_{10s'}, a_{101}, \dots, a_{10s''}) \\
&\quad \times \exp i(k_1\psi_{101} + \dots + k_{s''}\psi_{10s''}), \\
\frac{da_{10h}}{dt} &= \mu \sum{}' \Phi^{(1)}_{s'+h, k_1, \dots, k_{s''}, l_1, \dots, l_q}(\xi_{101}, \dots, \xi_{10s'}, a_{101}, \dots, a_{10s''}) \\
&\quad \times \exp i(k_1\psi_{101} + \dots + k_{s''}\psi_{10s''}), \\
\frac{du_{10h}}{dt} &= \mu \sum{}' \Phi^{(2)}_{s'+h, k_1, \dots, k_{s''}, l_1, \dots, l_q}(\xi_{101}, \dots, \xi_{10s'}, a_{101}, \dots, a_{10s''}) \\
&\quad \times \exp i(k_1\psi_{101} + \dots + k_{s''}\psi_{10s''}),
\end{aligned} \tag{15}$$

where

$$\psi_{10h} = u_{10h} - \omega_h t.$$

In formulas (15) the summation extends over values $k_1, \dots, k_{s''}, l_1, \dots, l_q$ satisfying condition (9). Returning, with the aid of the first formula (3), to the former variables $x_1, \dots, x_n$, we obtain, in the first approximation, the solution of the system of equations (1)

$$x_j(t) = \sum_{\sigma=1}^{s'} v_{j\sigma}\xi_{10\sigma} + \sum_{h=1}^{s''} N_{jh} a_{10h} \cos(u_{10h} + \gamma_{jh}). \tag{16}$$

Received
13 II 1963

CITED LITERATURE

1. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1958.

2. B. V. Bulgakov, *Applied Mathematics and Mechanics*, 10, no. 2 (1946).

Note: Figure translations are in progress. See original paper for figures.