B. Pasynkov

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Abstract

Full Text

B. Pasynkov

On a Generalization of Topological Products

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All spaces considered below are assumed to be T_1-spaces, and all mappings continuous. In the present note one important special case of the new concept of a local topological product, generalizing the concept of a topological product, will be introduced and considered, and applications of the new concept will be given. The concept of a local product will be considered in full in a subsequent note.

I. **Definition 1** (basic). Let there be given a space X_0—the base, its open subset ${}_\alpha O_0$, and a space Z_α—the fiber, consisting of two isolated points. The **local product** $X_\alpha = P(X_0, Z_\alpha, {}_\alpha O_0)$ of the base X_0 by the fiber Z_α relative to the open set ${}_\alpha O_0$ is defined to be the space of such a decomposition $\bar{g}_\alpha$ of the topological product

$$X_0 \times Z_\alpha = \{(x_0, z_\alpha) : x_0 \in X_0,\ z_\alpha \in Z_\alpha\},$$

whose elements are: 1) the individual points $(x_0, z_\alpha) \in X_0 \times Z_\alpha$, if $x_0 \in {}_\alpha O_0$, and 2) the sets

$$F_{x_0} = \{(x_0, z_\alpha) : z_\alpha \in Z_\alpha\},$$

if $x_0 \in X_0 \setminus {}_\alpha O_0$.*

Obviously, the local product coincides with the topological product $X_0 \times Z_\alpha$ if ${}_\alpha O_0 \equiv X_0$.

Denote by p_0 the projection of the product $X_0 \times Z_\alpha$ onto the factor X_0. Then there is uniquely defined a mapping

$${}^\alpha_0\mathfrak{F} : X_\alpha \to X_0,$$

satisfying the relation

$$p_0 = {}^\alpha_0\mathfrak{F} \cdot g_\alpha,$$

where g_α is the natural mapping of the space $X_0 \times Z_\alpha$ onto the decomposition space X_α.

It turns out that the mapping ${}^\alpha_0\mathfrak{F}$ is open, closed, two-to-one, piecewise topological (1), and decomposing (2). If the space X_0 is respectively Hausdorff, (fully) regular, (perfectly) normal, (locally) bicompact, finally compact, (weakly, strongly) paracompact, then the local product

$$X_\alpha = P(X_0, {}_\alpha O_0)$$

will have the same property. Moreover,

$$\operatorname{ind} X_\alpha = \operatorname{ind} X_0;$$

if X_0 is normal, then

$$\dim X_\alpha = \dim X_0;$$

if X_0 is perfectly normal, then

$$\operatorname{Ind} X_\alpha = \operatorname{Ind} X_0.$$

Definition 2. Suppose that in the space X_0 a finite ordered system of open sets

$${}_{\alpha_1}O_0, \dots, {}_{\alpha_s}O_0$$

is given. According to Definition 1 we construct the local product

$$X_{\alpha_1} = P(X_0, {}_{\alpha_1}O_0)$$

and the mapping

$${}^{\alpha_1}_{0}\mathfrak{F}.$$

Assuming that the local products

$$X_{(\alpha_1 \dots \alpha_l)} = P(X_0, {}_{\alpha_1}O_0, \dots, {}_{\alpha_l}O_0),$$

$l \leq s-1$, and the mappings

$${}^{(\alpha_1 \dots \alpha_l)}_{0}\mathfrak{F} : X_{(\alpha_1 \dots \alpha_l)} \to X_0, \qquad {}^{(\alpha_1 \dots \alpha_l)}_{(\alpha_1 \dots \alpha_k)}\mathfrak{F} : X_{(\alpha_1 \dots \alpha_l)} \to X_{(\alpha_1 \dots \alpha_k)},$$

$k \leq l \leq s-1$, have already been constructed, we put

$$X_{(\alpha_1 \dots \alpha_s)} = P(X_0, {}_{\alpha_1}O_0, \dots, {}_{\alpha_s}O_0)$$

equal to

$$P(X_{(\alpha_1 \dots \alpha_{s-1})}, ({}^{(\alpha_1 \dots \alpha_{s-1})}_{0}\mathfrak{F})^{-1}({}_{\alpha_s}O_0)),$$

and the mappings

$${}^{(\alpha_1 \dots \alpha_s)}_{0}\mathfrak{F} : X_{(\alpha_1 \dots \alpha_s)} \to X_0$$

and

$${}^{(\alpha_1 \dots \alpha_s)}_{(\alpha_1 \dots \alpha_k)}\mathfrak{F} : X_{(\alpha_1 \dots \alpha_s)} \to X_{(\alpha_1 \dots \alpha_k)}$$

are set equal respectively to

$${}^{(\alpha_1 \dots \alpha_{s-1})}_{0}\mathfrak{F} \cdot {}^{(\alpha_1 \dots \alpha_s)}_{(\alpha_1 \dots \alpha_{s-1})}\mathfrak{F}$$

and

$${}^{(\alpha_1 \dots \alpha_{s-1})}_{(\alpha_1 \dots \alpha_k)}\mathfrak{F} \cdot {}^{(\alpha_1 \dots \alpha_s)}_{(\alpha_1 \dots \alpha_{s-1})}\mathfrak{F},$$

where the mapping

$${}^{(\alpha_1\dots\alpha_s)}_{(\alpha_1\dots\alpha_{s-1})}\mathfrak{F} : X_{(\alpha_1\dots\alpha_s)} \to X_{(\alpha_1\dots\alpha_{s-1})}$$

is constructed in accordance with Definition 1.

It turns out that one may regard

$$X_{(\alpha_1\dots\alpha_s)} \equiv X_{(\alpha_{i_1}\dots\alpha_{i_s})},$$

where $(\alpha_{i_1}\dots\alpha_{i_s})$– an arbitrary permutation of the set $(\alpha_1\dots\alpha_s)$; therefore everywhere in what follows the sets of indices α are understood up to all possible permutations.

Definition 3. Suppose now that in the space X_0 an arbitrary system $v = \{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, of open sets ${}_\alpha O_0$ is given. It can be shown that the system of all finite local products

$$X_{(\alpha_1\dots\alpha_s)} = P\left(X_0, \{{}_{\alpha_i}O_0\},\ i = 1, \dots, s\right),$$

connected by the mappings ${}^{(\alpha_1\dots\alpha_s)}_{(\alpha_1\dots\alpha_k)}\mathfrak{d}$ for $\{\alpha_1, \dots, \alpha_k\} \subseteq \{\alpha_1, \dots, \alpha_s\}$, forms an inverse spectrum

$$S = \left\{X_{(\alpha_1\dots\alpha_s)},\ {}^{(\alpha_1\dots\alpha_s)}_{(\alpha_1\dots\alpha_k)}\mathfrak{d}\right\}, \quad \alpha \in \mathfrak{A},$$

which we shall call conjugate to the system v. The limit $X_{\{\alpha\}}$ of this spectrum we take as the local product $P\left(X_0, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A}\right)$. It is clear that if ${}_\alpha O_0 \equiv X_0$ for all $\alpha \in \mathfrak{A}$, then

$$X_{\{\alpha\}} \equiv X_0 \times \prod_\alpha Z_\alpha.$$

The projections ${}^{(\alpha_1\dots\alpha_s)}_{0}\mathfrak{d}$ and ${}^{(\alpha_1\dots\alpha_s)}_{(\alpha_1\dots\alpha_k)}\mathfrak{d}$ of the spectrum S turn out to be finite-to-one, piecewise-topological, open, closed, and decomposing mappings. The projections $\mathfrak{d}_{(\alpha_1\dots\alpha_s)} : X_{\{\alpha\}} \to X_{(\alpha_1\dots\alpha_s)}$ and $\mathfrak{d}_0 : X_{\{\alpha\}} \to X_0$ turn out to be open, closed, bicompact, zero-dimensional, and decomposing mappings. If the space X_0 is respectively Hausdorff, (completely) regular, (locally) bicompact, finally compact, (weakly, strongly) paracompact, then so is $X_{\{\alpha\}}$. Moreover,

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* In an analogous way the local product is also defined when the fiber Z_α is any bicompact space. If Z_α is not a bicompact space, then the definition of a local product is given in a somewhat different form. Since everywhere in this note the fiber Z_α is a simple two-point set, we shall omit the designation of the fiber.

$$\operatorname{ind} X_{\{\alpha\}} = \operatorname{ind} X_0;$$

if X_0 is strongly paracompact, then

$$\dim X_{\{\alpha\}} = \dim X_0;$$

if X_0 is a perfectly normal bicompactum, then

$$\operatorname{Ind} X_{\{\alpha\}} = \operatorname{Ind} X_0.$$

Finally,

$$w\left(X_{\{\alpha\}}\right) \leq \max\left(w(X_0), m(\mathfrak{A})\right)^* .$$

II. All the new universal spaces mentioned in the theorems of note (1) are local products; in particular, the bicompacta $P^{n\tau}$ are local products over n-dimensional tori C^n (i.e. C^n serve as bases for them). It turns out that the Menger universal curve M^1, which is not well suited to the topological product, is a local product over the interval I^1.

Theorem 1. *If in the n-dimensional cube I^n one takes such a countable base $v = \{_{nO}0\}$, $n = 1, 2, ...$, that any pair of points x_0' and x_0'' of I^n is contained in only a finite number of elements of the base v, then the local product*

$$P\left(I^n, \{_{nO}0\}, n = 1, 2, ...\right) = P^n :$$

1) will be a universal space for all n-dimensional metric spaces with a countable base; 2) will be locally connected; 3) will not have locally separating points; 4) will be locally universal for all n-dimensional metric spaces with a countable base, i.e. no open subset of P^n is embeddable in $2n$-dimensional Euclidean space.

From Theorem 1 and from Anderson' s results (^{3}) it follows:

Theorem 2. *The Menger universal curve M^1 is P^1, i.e. M^1 is the local product $P\left(I^1, \{_{nO}0\}, n = 1, 2, ...\right)$, where the system $\{_{nO}0\}$ is such a countable base of the interval I^1 that any two of its points are contained in only a finite number of elements of this base.*

III. As with topological products, local products are closely connected with the construction of universal spaces. It was already noted above that, for example, the bicompacta $P^{n\tau}$ are local products. If in the n-dimensional cube I^n one takes the same system of open sets $_{nO}0$, $n = 1, 2, ...$, as in Theorem 1, and takes the local product

$$P\left(I^n, \{Z_n\}, \{_{nO}0\}, n = 1, 2, ...\right),$$

where all layers Z_n are not two-point spaces but D^τ, i.e. topological products of τ copies of a two-point space, then we obtain an n-dimensional analogue of the Menger curve of weight τ. The bicompacta obtained will be universal spaces for the same spaces as the bicompacta $P^{n\tau}$ from (^{1}), and will have the same properties as $P^{n\tau}$, with the exception, perhaps, homogeneity. The principal role in the construction of new universal spaces is played by

Theorem 3. In order that the space Y have a homeomorphic mapping f into the local product $P(X_0, \{_\alpha O_0\}, \alpha \in \mathfrak{A})$, it is necessary and sufficient that the space Y have such a refining mapping $f_0 : Y \to X_0$, for which there exists a base (see (1), p. 1219) consisting of the union of two systems of sets: a) $\{_\alpha O_0, {}_\alpha O', {}_\alpha O''\}, \alpha \in \mathfrak{A}$, b) $\{_\beta V_0, {}_\beta V' = f_0^{-1}({}_\beta V_0), {}_\beta V'' = \Lambda\}, \beta \in \mathfrak{B}$, where the system $\{_\beta V_0\}, \beta \in \mathfrak{B}$, is a base of the space X_0.

Theorems 2, 8, and 10 of (1) can be supplemented by the following assertion.

Theorem 4. Among all: a) completely regular, b) (weakly, strongly) paracompact, c) finally compact, d) bicompact spaces X_0, possessing a refining mapping f_0 with $cw(f_0) \leq \tau$ (see (1)) into: a) a completely regular, b) a (weakly, strongly) paracompact, c) a finally compact, d) a bicompact space X_0 with $w(X_0) \leq \tau$, there exists a universal space X of weight $\leq \tau$, and X is a local product $P(X_0, \{_\alpha O_0\}, \alpha \in \mathfrak{A})$ with respect to some system $\{_\alpha O_0\}, \alpha \in \mathfrak{A}$.

IV. The principal role in applications of local products is played by

Theorem 5. Suppose a mapping $f_0 : Y_0 \to X_0$ is given and in the space X_0 there is a system $\nu = \{_\alpha O_0\}, \alpha \in \mathfrak{A}$, of open sets ${}_\alpha O_0$, while in the space Y_0 there is a system $\mu = \{_\alpha V_0\}, \alpha \in \mathfrak{A}$, of open sets ${}_\alpha V_0$, with ${}_\alpha V_0 \supset f_0^{-1}({}_\alpha O_0)$. It turns out that there exists a mapping

$$f : Y_{\{\alpha\}} = P(Y_0, \{_\alpha V_0\}, \alpha \in \mathfrak{A}) \to X_{\{\alpha\}} = P(X_0, \{_\alpha O_0\}, \alpha \in \mathfrak{A})$$

such that: 1) $f_0 \cdot \pi_0 = \mathfrak{E}_0 \cdot f$, where π_0 and $\mathfrak{E}_0$ (defined earlier) are the mappings of the local products $Y_{\{\alpha\}}$ and $X_{\{\alpha\}}$ onto the bases Y_0 and X_0; 2) $f \cdot \pi_0^{-1} = \mathfrak{E}_0^{-1} \cdot f_0$, i.e. $f(Y_{\{\alpha\}}) = X_{\{\alpha\}}$ when $f_0(Y_0) = X_0$; 3) $\operatorname{ind} f = \operatorname{ind} f_0$; 4) if the mapping f_0 is closed and bicompact, then the mapping f is closed and bicompact; 5) if ${}_\alpha V_0 = f_0^{-1}({}_\alpha O_0)$ for all $\alpha \in \mathfrak{A}$ and the mapping f_0 is open, then the mapping f is open as well; 6) if ${}_\alpha V_0 = f_0^{-1}({}_\alpha O_0)$ for all $\alpha \in \mathfrak{A}$ and $\mathfrak{E}_0(x) = x_0$, then the sets $f_0^{-1}(x_0)$ and $f^{-1}(x)$ are homeomorphic to one another by means of the mapping π_0. In particular, if the mapping f_0 is: a) $(k+1)$-fold, $k = 1, 2, ...$, b) finite

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* $w(X)$ denotes the weight of the space X, and $m(A)$ the cardinality of the set A.

(countably) multiple, c) n-dimensional in the sense dim(ind), $n = 0, 1, 2, \ldots$, then the mapping f will be the same; if all sets $f_0^{-1}(x_0)$, $x_0 \in X_0$, are homeomorphic to one another, then also all sets $f^{-1}(x)$, $x \in X_{\{\alpha\}}$, are homeomorphic to one another.

Corollary 1. If X_0 is a dyadic bicompactum, then any local product over X_0 is a dyadic bicompactum.

As an application of Theorem 5 we obtain the following two assertions.

Theorem 6. If the space X_0 is a $(k+1)$-fold and closed image of a completely regular space Y_0 with $\operatorname{ind} Y_0 = 0$, then any local product $X_{\{\alpha\}} = P(X_0, \{_\alpha O_0\}, \alpha \in \mathfrak{A})$ over X_0 will be a $(k+1)$-fold and closed image of a completely regular space $Y_{\{\alpha\}}$ with $\operatorname{ind} Y_{\{\alpha\}} = 0$, and if the space Y_0: a) is bicompact, b) finally compact, c) (weakly, strongly) paracompact, then $Y_{\{\alpha\}}$ will be the same; i.e. if X_0 is a perfectly k-dimensional paracompactum (4,5), then so are all local products $X_{\{\alpha\}}$ over X_0, i.e. in this case $\dim X_{\{\alpha\}} = \operatorname{ind} X_{\{\alpha\}} = \operatorname{Ind} X_{\{\alpha\}} = k$.

Theorem 7. If the space A has a refining mapping into the space X_0, which is a $(k+1)$-fold and closed image of a completely regular space Y_0 with $\operatorname{ind} Y_0 = 0$*, then the space A will also be a $(k+1)$-fold and closed image of a completely regular space B with $\operatorname{ind} B = 0$, and if the space A is (weakly, strongly) paracompact (finally compact, bicompact), then the space space B, i.e., if A is strongly paracompact, then it will be perfectly k-dimensional, i.e. $\dim A = \operatorname{ind} A = \operatorname{Ind} A$.

V. Theorem 3 from $^{(1)}$ is also an application of Theorem 5. The results of Theorem 3 from $^{(1)}$ can be strengthened:

Theorem 8. 1) Every compactum is an open and zero-dimensional image of a one-dimensional compactum.

2) Every bicompactum X of weight τ is an open and zero-dimensional image of a one-dimensional, in the sense of dim, bicompactum Y of weight τ. If X has a zero-dimensional mapping onto a compactum, then Y also has a zero-dimensional mapping onto a compactum, and then $\dim Y = \operatorname{ind} Y = \operatorname{Ind} Y = 1$.

3) Every: a) completely regular, b) (weakly, strongly) paracompact, c) finally compact space X of weight τ is an open, closed, bicompact, and zero-dimensional image of: a) a completely regular, b) a (weakly, strongly) paracompact space Y of weight τ, which is a subspace of a one-dimensional, in the sense of dim, bicompactum, i.e. if X is a strongly paracompact (finally compact) space, then $\dim Y = 1$. If the space X has a refining mapping onto a metric space, then the space Y also has a refining mapping

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* For example, the space Y_0 may be a metric space with $\dim Y_0 = k$.

onto a metric space, and in this case $\operatorname{ind} Y = 1$, i.e. if X is a strongly paracompact space, then $\dim Y = \operatorname{ind} Y = \operatorname{Ind} Y = 1$.

VI. Local products make it possible to represent refining mappings (in a certain sense) as superpositions of two-fold mappings:

Theorem 9. A mapping f_0 of a space Y onto a space Y_0 will be refining if and only if the space Y is an everywhere dense subset of the limit $\bar{Y}$ of such a spectrum $S_Y = \{Y_\alpha, \mathfrak{F}^{\beta}_{\alpha}\}$, $\alpha \in \mathfrak{A}$, that: 1) the projections $\mathfrak{F}^{\beta}_{\alpha}$ are finite-to-one, piecewise-topological, refining mappings "onto" ; 2) the space Y_0 is a minimal element in the spectrum S_Y, i.e. $0 < a$ for every index $\alpha \in \mathfrak{A}$; 3) the projection $\mathfrak{F}_0 : \bar{Y} \to Y_0$ coincides on the set $Y \subseteq \bar{Y}$ with the mapping f_0; 4) every projection $\mathfrak{F}^{\alpha}_0 : Y_\alpha \to Y_0$ is represented as a superposition of a finite number of two-fold projections $\mathfrak{F}^{\alpha}_{\alpha_1}, \mathfrak{F}^{\alpha_1}_{\alpha_2}, \dots, \mathfrak{F}^{\alpha_{s-1}}_{\alpha_s}, \mathfrak{F}^{\alpha_s}_{0}$, $s = s(\alpha)$; 5) then and only then $Y \equiv \bar{Y}$, when the mapping f_0 is bicompact.

In particular, Theorem 9 gives a characterization of zero-dimensional mappings of bicompacta and locally bicompact spaces. For example, for compacta we have the following theorem:

Theorem 10. A mapping f_0 of a compactum Φ onto a compactum Φ_0 is zero-dimensional if and only if Φ is the limit of a spectrum $S = \{\Phi_n, \mathfrak{F}^{m}_{n}\}$, $n = 0, 1, 2, \dots$, where each projection $\mathfrak{F}^{n+1}_{n}$, $n = 0, 1, 2, \dots$, is a two-fold and piecewise-topological mapping "onto" and $\mathfrak{F}_0 \equiv f_0$.

Theorem 11. On the local product $X_\alpha = P(X_0, \{{}_{\alpha}O_0\}, \alpha \in \mathfrak{A})$ there acts a bicompact zero-dimensional commutative group D^τ, which is the direct product of τ groups of the second order, where $\tau = m(\mathfrak{A})$, and the orbit space of the space $X_{\{\alpha\}}$ under the action of the group D^τ coincides with the space X_0, while the mapping $\mathfrak{F}_0 : X_{\{\alpha\}} \to X_0$ coincides with the natural mapping of $X_{\{\alpha\}}$ onto the orbit space. If the space X_0 is homogeneous, and the system $\nu = \{{}_{\alpha}O_0\}, \alpha \in \mathfrak{A}$, is such that together with the set ${}_{\alpha}O_0$ the system ν also contains all sets $g({}_{\alpha}O_0)$, $g \in G$, where G is a group acting on X_0, then the space $X_{\{\alpha\}}$ is also homogeneous.

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