ABSOLUTES OF HAUSDORFF SPACES

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Abstract

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ABSOLUTES OF HAUSDORFF SPACES

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In the present paper, for the class $\mathfrak{K}_2$ of all Hausdorff spaces and for the class $\mathfrak{F}$ of all irreducible θ-continuous closed* mappings, a theory of absolutes** is constructed (we note that for regular spaces all θ-continuous mappings are continuous, and for bicompacta all continuous mappings are perfect (see the footnote)). Namely: in the class $\mathfrak{K}_2$ we single out the subclass $\mathfrak{A}$ of all spaces X possessing the following property A:

A. *Every irreducible θ-continuous closed mapping of some space T onto the given space X is a homeomorphism.*

In other words, the spaces of the class $\mathfrak{A}$—we shall call them **absolutes**—are maximal with respect to the mappings of the class $\mathfrak{F}$ under consideration. These absolutes have a number of interesting properties (see Theorem 3 and Lemma 2). For every space X there exists "its own, uniquely determined, absolute $\widetilde{X}$" in the following precise sense:

Theorem 1. *For every Hausdorff space X there exists an absolute, irreducibly perfectly and θ-continuously mapped onto X; moreover, if the absolutes $\widetilde{X}$ and $\dot{X}$ are mapped onto X by means of almost-perfect*** (in particular, perfect) mappings $\tilde{g}$ and $\dot{g}$ of the class $\mathfrak{F}$, then there is a homeomorphism g such that $\tilde{g} = \dot{g}g$.*

It seems to us that the class of Hausdorff spaces is a sufficiently general and at the same time natural class within whose framework the theory set out can be considered without special complications, and also that the construction proposed here for obtaining the absolute $\widetilde{X}$ is the simplest of all those known at present****. We note also that absolutes are closely connected with the extensions of S. V. Fomin and E. Čech (see the corollaries of Theorem 3).

Apparently, Gleason ([2]) was the first to single out the class of absolutes in the class of all bicompacta and proved Theorem 1 for them. Among his other results we mention that the property of extremal disconnectedness is equivalent to property A for bicompacta. In fact this is true for arbitrary regular spaces—

* Let g be a mapping of a space X into a space Y. It is called **θ-continuous** if, for every point x of X and for every neighborhood Oy of the point $y = gx$, there exists a neighborhood Ox such that $g\overline{Ox} \subset \overline{Oy}$ (the bar denotes closure). It is called **closed** if the image $g\Phi$ of every closed set Φ in X is closed in gX.

It is called **irreducible** if, for every closed set Φ in X distinct from X, we have $g\Phi \neq gX$. It is called **bicompact** if the complete preimage $g^{-1}y$ of each point y of the space Y is bicompact. Finally, it is called **perfect** if it is both closed and bicompact.

** This work arose as a result of a talk by P. S. Aleksandrov concerning the absolutes of V. Ponomarev (see ([6])).

*** A mapping g of a space Y into a space X is called **almost-perfect** if, for each point x of X and for any family of open sets H_α covering the preimage $g^{-1}x$, there exists a neighborhood Ox of it such that $\operatorname{Int} g^{-1}\overline{Ox}$ is contained in the union of a finite number of sets $\overline{H}_\alpha$. Every irreducible perfect mapping is almost-perfect (see Theorem 4).

**** While this paper was being written, Yu. M. Smirnov suggested to me another construction for obtaining the absolute, and after my report a number of the most diverse constructions for obtaining the absolute were indicated by Ju. Flachsmeyer, V. I. Ponomarev ([9]), A. Pełczyński (A. Pełczynski), and B. Pasynkov.

spaces. Independently of Gleason, by means of an entirely different construction, V. I. Ponomarev ([6]) singled out a class of absolutes in the class of all paracompacta (relative to the class of all irreducible continuous perfect* mappings) and proved Theorem 1 for them.

Construction of the absolute. Let X be an arbitrary Hausdorff space. An **end** (ultrafilter) of the space X is any maximal centered** system of its open sets (no additional conditions!). We shall call an end **proper** if it contains all neighborhoods of some point of the space X. For each open subset Γ of X, denote by $O\Gamma$ the set of all ends containing Γ as an element.

Properties of the operator O: 1) $O(\Gamma \cap H) = O\Gamma \cap OH$; 2) if $\Gamma \subset \overline{H}$, then $O\Gamma \subset OH$; 3) $O(\Gamma \cup H) = O\Gamma \cup OH$; 4) $O(\Gamma \setminus \overline{H}) = O\Gamma \setminus OH$.

The set BX of all ends of the space X has the following natural topology: as a base of open sets we take the collection of all sets of the form $O\Gamma$. This space BX we shall call the **hyperabsolute**.***

Theorem 2. *The subspace $\widetilde{X}$ of the hyperabsolute BX, consisting of all proper ends, is the absolute of the space X***.*

Lemma 1. *The spaces X and BX are Hausdorff.*

In what follows denote by α the natural mapping which assigns to each proper end $\tilde{x}$ the (unique–by virtue of the Hausdorffness of the space X) point x all of whose neighborhoods the end $\tilde{x}$ contains.

Properties of the operator O: 5) $\overline{O\Gamma} = O\Gamma$; 6) $\alpha^{-1}\Gamma \subset O\Gamma \cap \widetilde{X} \subset \alpha^{-1}\overline{\Gamma}$; 7) if G is open in BX, then $\overline{G} = O \operatorname{Int} \alpha(G \cap \widetilde{X})$; 8) $G \cap \widetilde{X} = \overline{G}$.

Lemma 2. *BX and $\widetilde{X}$ are extremally disconnected*****, BX is a zero-dimensional bicompact extension of the space $\widetilde{X}$.*

Indeed, if $\{O\Gamma_{\alpha_i}\}$ is a cover of the space BX containing no finite subcover, then the system of sets

$$O\left(X \setminus \bigcup_i \overline{\Gamma}_{\alpha_i}\right) = BX \setminus \bigcup_i O\Gamma_{\alpha_i}$$

is centered. The system of sets

$$X \setminus \bigcup_i \overline{\Gamma}_{\alpha_i}$$

is also centered. Extending it to an end $\tilde{x}$ and taking the set $O\Gamma_a$ containing $\tilde{x}$, we obtain a contradiction to property 1). Hence, BX is bicompact. The rest follows easily from properties 6)–8).

Theorem 3. *For every H-closed extension hX of the space X, the mapping α extends in a unique way to an irreducible θ-continuous mapping α_h of the bicompact BX onto hX, and moreover*

$$\alpha_h^{-1}X = \widetilde{X}.$$

Indeed, let $\tilde{x} \in BX$, and $x \in hX$. Put $\alpha_h\tilde{x} = x$ if for every neighborhood Ox of the point x we have $X \cap Ox \in \tilde{x}$. It is clear that if $x \in X$, then $\alpha_h^{-1}x = \alpha^{-1}x$, and if $x \notin X$, then $\alpha_h^{-1}x \subset BX \setminus \widetilde{X}$. Suppose that $\tilde{x} \notin \alpha_h^{-1}(hX)$. Then for each x there exists a neighborhood Ox such that $\Gamma_x = X \cap Ox \notin \tilde{x}$. By virtue of H-closedness, there is a finite number of such sets $\Gamma_i = \Gamma_{x_i}$,

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* In fact, the definitions of the absolute by means of perfect mappings and by means of closed mappings (property A) are equivalent; see Theorem 3.

** This means that any two of its elements have nonempty intersection, and, in addition, every open set intersecting each of its elements belongs to this system (see (1)).

*** The space BX was obtained in a somewhat different way and for other purposes by S. V. Fomin ([7]).

**** The first assertion of Theorem 1 is an immediate consequence of Theorem 2 and Theorem 4.

***** A space is called **extremally disconnected** if the closure of every open set in it is open.

that $X = \bigcup \overline{X}\Gamma_i$. But then $B = \bigcup O\Gamma_i$ (properties 2), 4)). Hence, $\tilde{x} \in O\Gamma_i$ for some i, which cannot be. Thus, a_h is defined on all of BX. Further, if Φ

is closed and does not coincide with BX, then there is a nonempty open set Γ such that $O\Gamma \subseteq BX \setminus \Phi$. Then

$$a_h^{-1}\Gamma \cap \Phi = \alpha^{-1}\Gamma \cap \Phi = \emptyset^*.$$

Hence a_h is irreducible. θ-continuity follows easily from the following inclusions:

$$a_h O(X \cap Ox) = a_h O(X \cap Ox) \subseteq \overline{X \cap Ox}^h = \overline{Ox}^{h} {}^{**}$$

(see properties 1) and 5)). The theorem is proved.

Corollary 1. The mappings α and a_h are irreducible, perfect, and θ-continuous***.

Corollary 2. A Hausdorff space is H-closed if and only if it is the image of some (zero-dimensional) bicompactum under an (irreducible) θ-continuous mapping****.

"The absolute $\widetilde{X}$." For any mapping g of a set Y into a set X, consider the operator O_g, which assigns to each subset A of the set Y the following set:

$$O_g A = gY \setminus g(X \setminus A) = \mathcal{E}\{x : \emptyset \neq g^{-1}x \subseteq A\}.$$

Properties of the operator O_g: 1′) $O_g(A \cap B) = O_{gA} \cap O_{gB}$; 2′) if $A \subseteq B$, then $O_{gA} \subseteq O_{gB}$; 3′) $O_g(A \cup B) \supseteq O_{gA} \cup O_{gB}$, 4′) $O_g(A \setminus B) \subseteq O_{gA} \setminus O_{gB}$.

Lemma 3. A mapping g of a space Y into a set X is irreducible if and only if, for every nonempty open set H in Y, the set O_{gH} is nonempty, and moreover

$$g^{-1}O_{gH} = \overline{H}.$$

Theorem 4. The following properties of a Hausdorff space X are equivalent:
a) being an absolute*****;
a$_c$) every irreducible θ-continuous perfect mapping of some space Y onto X is a homeomorphism;
b) the mapping α is a homeomorphism;
c) being homeomorphic to the space $\widetilde{X}$;
d) being regular and extremally disconnected.

In view of the preceding, here we need only the following

Lemma 4. Every irreducible θ-continuous and closed mapping g of a space X onto an extremally disconnected space Y is one-to-one******.

Corollary 1. The spaces $\widetilde{X}$ and BX are absolutes.

Corollary 2. Always

$$\widetilde{\widetilde{X}} = \widetilde{X} \quad \text{and} \quad BBX = B\widetilde{X} = \widetilde{BX} = BX.$$

Corollary 3. If X is completely regular, then

$$\beta\widetilde{X} = \beta\widetilde{X} = BX$$

(βX is the Čech extension*******).

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* $\emptyset$ is the empty set.

** $\overline{M}^h$ is the closure in hX of the set M.

*** It is easy to see that these mappings are continuous if and only if the corresponding spaces X and hX are regular. Since there exist regular but not completely regular spaces, it follows at once that the property of complete regularity is not preserved under continuous perfect mappings ([3]), even if this mapping is irreducible and the image space is extremally disconnected. Hence we immediately obtain a generalization of V. I. Ponomarev's theorem ([5]) stating that every normal space is the image of an inductive-zero-dimensional space under an irreducible continuous and perfect mapping, to arbitrary regular spaces.

**** The sufficiency of this condition in the strengthened form (the image of every H-closed space under a θ-continuous mapping is also H-closed) was proved by S. V. Fomin ([8]).

***** It is easy to see that a space with the first axiom of countability (for example, a metric space) is an absolute if and only if it is discrete.

****** If g is continuous or if Y is regular, then g is a homeomorphism. Therefore we obtain various characterizations a', a'_c, b', and c' of extremal disconnectedness, if in the corresponding conditions of Theorem 4 we replace the word "homeomorphism" by "θ-homeomorphism," understanding by this a θ-continuous one-to-one closed mapping.

******* For other constructions of H-closed extensions (see ([4], [5], [7], [8])), the first equality (commutativity) is true only when this construction, applied to the absolute, yields the Čech extension. For a noncompact space X of category X, the Katětov extension τX is not bicompact, and therefore

$$\tau\widetilde{X} \neq \widetilde{\tau X}$$

(see ([4])).

For the proof we need corollaries of Theorems 1 and 2, as well as

Lemma 5. *The bicompact extension bX of the absolute X is an absolute if and only if $bX = \beta X$.*

Corollary 4. *If X is dense in eX, then $BeX = BX$.*

Corollary 5. *If Γ is open and regular $*$ or dense in X, then $\alpha^{-1}\Gamma = \tilde{\Gamma}$; if $N_h = hX \setminus X$ is nowhere dense in hX, then $\alpha^{-1}N_h = \widetilde{hX} \setminus \widetilde{X}$; if, moreover, hX is H-closed, then $\alpha^{-1}N_h = \beta\widetilde{X} \setminus \widetilde{X}$; if, in addition, X is an absolute (in particular, discrete), then $\alpha^{-1}N_h = \beta X \setminus X$ $**$.*

"Uniqueness of the absolute X and extension of mappings." Theorem 5. *For any irreducible θ-continuous closed mapping g of a space Y onto a space X, there exists a homeomorphic mapping $\tilde{g}$ of the hyperabsolute BY onto the hyperabsolute BX such that $\tilde{g}\widetilde{Y} \subseteq \widetilde{X}$ and $g\alpha = \alpha\tilde{g}$ on $\widetilde{Y}$; the mapping g is almost perfect if and only if $\tilde{g}^{-1}\widetilde{X} = \widetilde{Y}$ ***.*

Lemma 6. *If a θ-continuous mapping g of a space **** Y onto a space X is closed, and Γ is open in X, then*

$$O_g \operatorname{Int} g^{-1}\overline{\Gamma} = \operatorname{Int} \overline{\Gamma}.$$

Construction of the mapping $\tilde{g}$. If $\tilde{y} = \{H\} \in BY$, then the system $\{O_g H\}$ is open (closed) and centered (irreducibility). Let the image $\tilde{g}\tilde{y}$ consist of all such ends $\tilde{x}$ that $\{O_g H\} \subseteq \tilde{x}$. From Lemma 6 we obtain $\tilde{g}\tilde{y} = \tilde{x}$ if and only if $\operatorname{Int} g^{-1}\overline{\Gamma} \in y$ for every Γ from x. Therefore $\tilde{g}$ is one-to-one, continuous, and "onto." The remaining assertions require an additional proof.

Corollary 1. *The absolutes of spaces X and Y are homeomorphic if and only if there exists a space T that is mapped both onto X and onto Y by irreducible θ-continuous perfect mappings *****.*

Corollary 2. *If g is an irreducible continuous perfect mapping of a space Y onto a space X, where X and Y are completely regular, then the natural diagram for the mapping g and its extensions g_β (to the Čech extensions) and $\tilde{g}$ (to the hyperabsolutes) is commutative.*

Corollary 3. *Let a topological property P be preserved under continuous irreducible perfect mappings (in both directions) of regular spaces ******; then a regular space X has property P if and only if its absolute $\widetilde{X}$ has it.*

In conclusion, I express my sincere gratitude to Prof. Yu. M. Smirnov, under whose guidance this work was written.

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* This means that for every point x of Γ there is a neighborhood Ox such that $\overline{Ox} \subseteq \Gamma$.

** In the particular case when X is discrete, hX is bicompact and N_h consists of one point, we obtain a theorem of B. Efimov. In the general case the last equality is false: for X one must take the plane, and for hX, the plane X completed by one point to a sphere.

*** In particular, the last equality holds for perfect mappings.

**** The spaces in this work are Hausdorff.

***** This condition is easily formulated in terms of multivalued mappings ([6]).

****** For example, the properties of bicompactness, final compactness, paracompactness, countable paracompactness, local bicompactness, etc. (see ([3],[6])).

Note: Figure translations are in progress. See original paper for figures.