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Abstract

Full Text

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On Infinite-Dimensional Linear Groups

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1. A linear group is a subgroup of the group of invertible linear operators (automorphisms) of a vector space. If the space is finite-dimensional, then the corresponding group is called **finite-dimensional**, and in the case of an infinite-dimensional space the group is called **infinite-dimensional**. The following general problem naturally arises: to study infinite-dimensional groups which, in one sense or another, are close to finite-dimensional ones. Some classes of such groups are considered in the present note.

The following notation and definitions are used in the note: G is a vector space over a certain field P; Γ is a certain group of automorphisms of this space; $[\Gamma]$ is the linear envelope of Γ in the algebra of all linear operators of the space G; Σ is a certain subgroup of Γ having a finite system of generators.

A system of Γ-admissible subspaces of the space G

$$G_0 = 0 \subset \dots \subset G_\alpha \subset G_{\alpha+1} \subset \dots \subset G_\gamma = G$$

is called **stable relative to** Γ if, for every $\sigma \in \Gamma$ and every jump of the system, one has

$$G_{\alpha+1} \circ (\sigma - \varepsilon) \subset G_\alpha;$$

ε is the identity in Γ.

The group Γ is called **weakly stable** (or **stable**, or **externally nilpotent**) if in G there is a system of subspaces stable relative to Γ (respectively, either an ascending stable series, or a finite stable series). Γ is called **locally stable** if, for every Σ in G, there is a local system of Σ-admissible subspaces H_α such that in each H_α, Σ acts as a stable group. It can be shown that if Γ is locally stable, then all such H_α may be taken to be finite-dimensional. The group Γ is called **locally externally nilpotent** if every Σ is nilpotent relative to G. A pair of Γ-admissible subspaces A and B, $A \subset B$, of G is called a **compositional pair** if there are no other Γ-admissible subspaces between them. We shall denote by $\alpha(\Gamma)$ the set of all elements σ of Γ such that, for every compositional pair A, B, $A \subset B$, one has

$$B \circ (\sigma - \varepsilon) \subset A.$$

$\alpha(\Gamma)$ is a normal divisor in Γ, called the **external radical** of this group. By $\beta(\Gamma)$ is denoted the locally stable radical of the group Γ—the subgroup in Γ generated by all its locally stable normal divisors. $\gamma(\Gamma)$ is the subgroup in Γ generated by all locally externally nilpotent normal divisors; this subgroup is itself locally externally nilpotent. Some general relations among the three indicated radicals were considered in the note ([1]).

We now define finiteness conditions. We shall call the group Γ ***c*-finite** if in G there is a Γ-admissible finite-dimensional subspace H such that the induced representation of Γ relative to H is faithful; we shall call Γ ***b*-finite** if in G there is a local system of Γ-admissible finite-dimensional subspaces; we shall call Γ ***a*-co-**

finite, if $[\Gamma]$ is a finite-dimensional algebra. It is easy to see that a-finiteness is here the strongest finiteness condition. The notions of local c-finiteness, local b-finiteness, and local a-finiteness of the group Γ are now defined in the corresponding way.

2. From the general results of note ([1]) it follows that if Γ is a locally b-finite group, then $\alpha(\Gamma) = \beta(\Gamma)$, so that $\alpha(\Gamma)$ has a central system and is a locally stable group ([1]). We shall show that if Γ is locally c-finite, then $\alpha(\Gamma)$ is a locally nilpotent group, while $\beta(\Gamma)$ is a locally stable group and belongs to $\alpha(\Gamma)$.

Let Φ be a locally c-finite weakly stable group of automorphisms of the space G, and let Σ be a subgroup of Φ having a finite number of generators. Find in G a finite-dimensional Σ-admissible subspace H such that the pair (H, Σ) is exact. Since Σ is a weakly stable group and since H is finite-dimensional, Σ is nilpotent relative to H. In view of the fact that the pair (H, Σ) is exact, we conclude that Σ is nilpotent as an abstract group; hence Φ is a locally nilpotent group.

Taking into account that $\alpha(\Gamma)$ is a weakly stable group, in the case under consideration we obtain that $\alpha(\Gamma)$ is a locally nilpotent group. The same can be said about $\beta(\Gamma)$, since a locally stable group is always weakly stable. Let further $R(\Gamma)$ be the locally nilpotent radical of the group Γ. Since the radicals $\alpha(\Gamma)$ and $\beta(\Gamma)$, by what has been proved, belong to $R(\Gamma)$, we obtain

$$\alpha(\Gamma) = \alpha(R(\Gamma)) \quad \text{and} \quad \beta(\Gamma) = \beta(R(\Gamma)).$$

Keeping in mind, moreover, that a locally nilpotent group is a locally Noetherian group, on the basis of ([1]) we now conclude that $\beta(\Gamma)$ is a locally stable group belonging to $\alpha(\Gamma)$.

From known results concerning finite-dimensional linear groups it is easy to conclude that if Γ is a locally c-finite group, then the locally nilpotent radical $R(\Gamma)$ coincides with the set of all nil-elements of the group Γ. In addition, Γ has a locally solvable radical—the locally solvable normal divisor containing all locally solvable normal divisors of Γ. It is easy to verify that this radical is a radical in the sense of A. G. Kurosh in the class of locally c-finite linear groups ([2]).

3. In the case when the group Γ is locally a-finite, all three radicals $\alpha(\Gamma)$, $\beta(\Gamma)$, and $\gamma(\Gamma)$ coincide. The coincidence $\alpha(\Gamma) = \beta(\Gamma)$ follows from the preceding remarks, and the equality $\beta(\Gamma) = \gamma(\Gamma)$ is contained in the following proposition.

A locally stable group Γ is locally a-finite if and only if it is locally externally nilpotent.

Indeed, let Γ be a locally stable group, and suppose first that it is locally a-finite. Let Σ be a subgroup of Γ with a finite number of generators, and let n be the dimension of the algebra $[\Sigma]$. Take in G an arbitrary element g, and let H be the subspace in G consisting of all elements of the form $g \circ \varphi$, $\varphi \in [\Sigma]$. Then H is a Σ-admissible subspace, and its dimension does not exceed n. Since Γ is a locally stable group, Σ is nilpotent relative to H. The length of a stable series in H cannot be greater than n. This proves that for any $g \in G$ all higher commutators of the form $[g; \sigma_1, \sigma_2, \dots, \sigma_m]$ over all $\sigma_i \in \Sigma$ with length $m \geqslant n$ are transformed into zero in the space G. Hence it follows that in G there is a stable series relative to Σ of length $\leqslant n$. Thus Γ is a locally externally nilpotent group.

We prove the converse. Let Γ be locally externally nilpotent (by this, of course, it is said that Γ is locally stable). Take in Γ an arbitrary subgroup Σ with a finite number of generators $\sigma_1, \sigma_2, \dots, \sigma_k$. Since Σ is externally nilpotent, there exists an n such that the product of any n elements of the form $\sigma_i - \varepsilon$ is equal to zero, so that the product of any n elements of the form σ_i, $i = 1, 2, \dots, k$, is expressed linearly in terms of products of such

elements with a smaller number of factors. Hence it obviously follows that the subalgebra $[\Sigma]$ has finite dimension.

Starting from the equality $\alpha(\Gamma) = \gamma(\Gamma)$, analogously to how this is done in (1), one can obtain the following further result:

If the group Γ is locally a-finite, then the outer radical of this group is connected with the radical of the Lie algebra $[\Gamma]$ by the formula

$$\alpha(\Gamma) = \Gamma \cap (L([\Gamma]) + \varepsilon).$$

Let us prove the following assertion.

In an arbitrary linear group Γ, the product of an invariant locally a-finite subgroup Φ_1 and a locally a-finite subgroup Φ_2 is a locally a-finite subgroup.

First consider the case where Φ_1 and Φ_2 are a-finite. In this case, in Φ_1 and Φ_2 there are finite subsets, respectively X and Y, such that X linearly generates $[\Phi_1]$ and Y linearly generates $[\Phi_2]$. We may also assume that X and Y contain the identity of the group Γ. In this case the product of sets XY contains both X and Y. Denote by S the linear hull of the set XY. This hull is finite-dimensional and contains $[\Phi_1]$ and $[\Phi_2]$. We show that S is closed with respect to multiplication. Each element of $\Phi_1\Phi_2$ has the form $\varphi_1\varphi_2$, $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$. But φ_1 and φ_2 are expressed linearly in terms of elements of X and, respectively, Y, so that $\varphi_1\varphi_2$ is expressed linearly in terms of elements of XY. Consequently, the subgroup $\Phi_1\Phi_2$ belongs to S. Hence follows the closedness of S with respect to multiplication, as well as the fact that $\Phi_1\Phi_2$ is an a-finite group.

We pass to the general case. Let Φ_1 and Φ_2 be locally a-finite, and let Σ_2 be a subgroup with a finite number of generators in Φ_2. Since this subgroup is a-finite, in it there is a finite subset Y linearly generating the subalgebra $[\Sigma_2]$. Next, let X be an arbitrary finite subset in Φ_1. Denote by Z the set of all elements of the form $\sigma_2^{-1}\sigma\sigma_2$, $\sigma \in X$, $\sigma_2 \in \Sigma_2$. Since Z is contained in the linear hull of the finite set YXY, there is in Z a finite subset Z_0 such that the linear hull of Z_0 coincides with the linear hull of Z. We may assume that $X \subset Z_0$. Let Σ_1 be the subgroup in Φ_1 generated by the set Z, and let Σ_0 be generated by Z_0. It is clear that $[\Sigma_0] = [\Sigma_1]$, and, consequently, Σ_1 is an a-finite group. Since the subgroup Σ_1 is invariant with respect to Σ_2, the group $\Sigma_1\Sigma_2$ is a-finite. It is obvious that every subgroup with a finite number of generators from $\Phi_1\Phi_2$ is contained in some such $\Sigma_1\Sigma_2$, and the assertion is thereby proved.

From it follows the following theorem.

In an arbitrary linear group Γ, the intersection of all maximal locally a-finite subgroups is an invariant locally a-finite subgroup in Γ, containing all other such invariant subgroups.

This subgroup is naturally called the **locally a-finite radical**.

4. We now consider some properties of a locally nilpotent linear group. According to (3), in such a group Γ the radical $\beta(\Gamma)$ coincides with the set of all nilautomorphisms of the space G lying in Γ. It is also easy to show that $\gamma(\Gamma)$ is the totality of all $\sigma \in \Gamma$ for which the endomorphism $\sigma - \varepsilon$ is a nilpotent endomorphism.

We shall next prove the following proposition.

Let Γ be a locally nilpotent group of automorphisms of the vector space G, and let Φ be its invariant a-finite subgroup. Then, if Φ is weakly stable, this subgroup has a finite central series of subgroups relative to the whole group Γ.

For the proof we first note that from the weak stability and a-finiteness of Φ follows the outer nilpotence of this subgroup. Let the series

$$G = G^0 \supset G^1 \supset ... \supset G^n = 0$$

is a decreasing series of Φ-commutants of the space G. We shall prove the proposition by induction on the length of such a series. For length equal to 1 the assertion is obvious, and suppose that it has been proved for lengths $< n$. From this assumption it is easy to derive that if Σ is the Φ-centralizer of the subgroup G^1, then in Φ/Σ there is a finite central series in Γ/Σ. It remains to show that in Σ there is also a finite central series relative to Γ.

Denote by Σ^* the linear span of the set of all elements of the form $\sigma - \varepsilon$, $\sigma \in \Sigma$, and consider the mapping $\sigma \to \sigma^* = \sigma - \varepsilon \in \Sigma^*$. A simple verification shows that such a mapping is an isomorphic embedding of the group Σ into the vector space Σ^*. Denote by Σ' the image of Σ under this mapping in Σ^*. Next define a representation of the group Γ by automorphisms of the vector space Σ^*, putting $\eta \circ \gamma = \gamma^{-1}\eta\gamma$ for $\eta \in \Sigma^*$ and $\gamma \in \Gamma$. It is easy to see that then the subgroup $\Sigma' \subset \Sigma^*$ is Γ-admissible and that the mapping* defines an isomorphism of the group pair (Σ', Γ) and the internal pair (Σ, Γ). Since the group Γ is locally nilpotent, the group pair (Σ, Γ) is locally stable. Consequently, the pair (Σ', Γ), and along with it also the pair (Σ^*, Γ), is locally stable. Taking into account that the vector space Σ^* is finite-dimensional, by a known theorem we now conclude that Γ is nilpotent relative to Σ^*. But then Γ is nilpotent also relative to Σ, and this means that the required property of the subgroup Σ holds.

Let us note two consequences of this proposition.

By analogy with the abstract theory, we shall call a linear group Γ **locally a-normal** if in this group there is a local system of a-finite normal divisors.

If the group Γ is weakly stable and locally a-normal, then it has an increasing central series of length not greater than ω.

Another consequence is the following generalization of a theorem of Zassenhaus ([4,5]).

Let Γ be a locally nilpotent group of automorphisms of a vector space G, having an a-finite outer radical. Then, if in G there is a finite series of Γ-admissible subspaces, on all factors of which Γ acts as a nilpotent group, then the group Γ itself is nilpotent.

5. In all the cases considered above, the outer radical $a(\Gamma)$ possesses a central system (being a Z-group). We shall show that in the general case this radical need not necessarily be a Z-group.

Let G be a countable-dimensional vector space over the field of rational numbers with basis $e_1, e_2, \dots, e_n, \dots;\ e_0$, and let $\sigma, \varphi, \psi_1, \psi_2, \dots, \psi_n, \dots$ be linear operators of this space defined by the formulas:

$$\left.\begin{array}{lll} e_{2k} \circ \sigma = e_{2k} + \frac{1}{2^k} e_0; & e_{2k-1} \circ \sigma = e_{2k-1} + \frac{1}{2^{k-1}} e_0; & e_0 \circ \sigma = e_0; \\ e_{2k-1} \circ \varphi = e_{2k-1} + e_{2k} + e_{2k+1}; & e_{2k} \circ \varphi = e_{2k} + e_{2k+1}; & e_0 \circ \varphi = e_0; \end{array}\right\} k \geqslant 1;$$

$$e_i \circ \psi_j = e_i + e_{i+1} \text{ for } 1 \leqslant i < j; \qquad e_i \circ \psi_j = e_i \text{ for } i \geqslant j \text{ and } i = 0.$$

It is not difficult to verify that all these operators are automorphisms of the space G. Denote by Γ the group generated by them. One can show that $\Gamma = a(\Gamma)$.

On the other hand, from the easily verified relation $\varphi\sigma\varphi^{-1}\sigma^{-1} = \sigma$ it follows that the group Γ is not a Z-group.

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Note: Figure translations are in progress. See original paper for figures.