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Abstract

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A UNIQUENESS THEOREM FOR DIRICHLET SERIES

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The present note is devoted to the following problem. Let there be an entire function $F(z)$, representable by the Dirichlet series

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \tag{1}$$

absolutely convergent in the whole z-plane. In addition, suppose that

$$|F(x)| \le C, \qquad -\infty < x < \infty, \tag{2}$$

i.e., the Dirichlet series under consideration is bounded on the real axis. Denote

$$M_F(x) = \sum_{n=1}^{\infty} |a_n| e^{\lambda_n x}.$$

It is required to find conditions imposed on the growth of $M_F(x)$ under which there can exist a nonzero Dirichlet series satisfying condition (2).

We note that the limiting growth of $M_F(x)$ allowing the existence of a Dirichlet series bounded on the real axis depends on the behavior of the sequence $\{\lambda_n\}$; roughly speaking, the sparser this sequence is, the greater the growth of $M_F(x)$ must be.

It is also worth noting that the requirements on the growth of $M_F(x)$ can be replaced by conditions on the coefficients a_n. Indeed,

$$|a_n| \le \min_x \{M_F(x) e^{-\lambda_n x}\}.$$

The problem under consideration was posed by I. A. Evgrafov and was solved by him to a considerable extent in papers ([1,2]). He considered the case where $\lambda_n \sim n l(n)$, where $l(n)$ is a slowly increasing function.

The question naturally arises of finding the limiting growth for the case $\lambda_n \sim n^{1/q} l(n)$, $q > 1$ (for $q < 1$ the answer is trivial). In the present note we restrict ourselves to sequences of the form $\lambda_n \sim c n^{1/q}$, $q > 1$.

We note further that if $M_F(x) = e^{o(x^2)}$, then a nonzero Dirichlet series cannot be bounded on the real axis, whatever the sequence of its exponents λ_n may be. Indeed, since all λ_n are real, $|F(z)| \leq M_F(x)$. Hence, in the whole plane $F(z) = e^{o(z^2)}$. Moreover,

$$|F(x)| \leq C, \qquad |F(iy)| \leq M_F(0) = C_1.$$

Applying the classical Phragmén-Lindelöf theorem to the function $F(z)$ in each quadrant, we obtain that $|F(z)| < C_2$ in the whole plane, i.e., by Liouville' s theorem, $F(z) \equiv \text{const}$. But as $z \to -\infty$, evidently, $F(z) \to 0$. Consequently, $F(z) \equiv 0$.

Thus, it remains to restrict ourselves only to the condition $q < 2$. The case $q \geq 2$ requires special investigation.

Theorem 1. *Let there be a Dirichlet series of the form (1), absolutely convergent in the whole complex plane. Suppose that*

$$\lim_{n\to\infty} \lambda_n n^{-1/q} = c, \qquad 0 < c < \infty, \qquad 1 < q < 2. \tag{3}$$

If $M_F(x) \ll C_\varepsilon e^{(\alpha-\varepsilon)x^p}$, $\varepsilon > 0$ arbitrary, where p is determined by the equality $1/p + 1/q = 1$,

$$\alpha = c^p (q-1) q^{-p} \left(-\pi \operatorname{tg} \frac{\pi q}{2}\right)^{1-p},$$

then from the boundedness of the Dirichlet series on the real axis it follows that this series is identically equal to zero.

We shall prove this theorem according to the scheme proposed in ([1]). First we formulate several auxiliary assertions.

Lemma 1. *Let $F(z)$ be the Dirichlet series (1), bounded on the real axis. Denote*

$$G(z) = \int_0^\infty F(z+t) e^{-\alpha_1 t^p}\, dt, \qquad \alpha - \varepsilon < \alpha_1 < \alpha.$$

Then $G(z)$ is also a Dirichlet series, bounded on the real axis:

$$G(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n z},$$

where

$$c_n = a_n \int_0^\infty e^{-\alpha_1 t^p + \lambda_n t}\, dt.$$

The proof is obtained by substituting, in place of the integrand $F(z+t)$, its expression as a series and integrating term by term.

Lemma 2. *Let $H(z)$ be a Dirichlet series of the form (1), bounded on the real axis. Then its two-sided Laplace transform $\widetilde{H}(z)$ can be represented as the sum of two functions, namely*

$$\widetilde{H}(z) = \widetilde{H}_1(z) + \widetilde{H}_2(z),$$

where the function $\widetilde{H}_1(z)$ is regular and bounded in the half-plane $\operatorname{Re} z > 0$, while the function $\widetilde{H}_2(z)$ has the form

$$\widetilde{H}_2(z) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n - z}.$$

The proof is given in ([1]).

Lemma 3. Denote

$$E(z) = \int_0^\infty e^{-\alpha_1 t^p + zt}\, dt.$$

The equality holds

$$\widetilde{G}(z) = \widetilde{F}(z) E(z), \qquad \widetilde{F}(z) = \int_{-\infty}^\infty F(x) e^{-xz}\, dx.$$

Lemma 4. *For the function $E(z)$ the estimate holds*

$$\ln E(x) = (p-1)\alpha_1^{-\frac{1}{p-1}} p^{-q} x^q (1 + o(1)) \qquad (x \to +\infty).$$

Proof. The integral representing $E(x)$ is easily estimated by Laplace' s method.

Lemma 5. Denote

$$P(z) = \prod_{n=1}^\infty \left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right).$$

For the function $P(z)$ the estimate holds

$$\overline{\lim_{r\to\infty}}\, r^{-q}\ln\left|P(re^{i\varphi})\right| = \pi c^{-q}|\sin q\varphi|, \qquad |\varphi| \le \frac{\pi}{q}.$$

The proof is obvious.

Proof of Theorem 1. Introduce the function

$$\chi(z) = \widetilde{F}(z)P(z). \tag{4}$$

According to Lemma 3, it can be represented in the form

$$\chi(z) = \frac{\widetilde{G}(z)}{E(z)}P(z). \tag{5}$$

Our goal is to estimate this function on the imaginary axis, which we shall obtain from the estimate of its indicator in the right half-plane.

First let us show that the function $\widetilde{G}(z)P(z)$ on the real axis is equal to $e^{o(x^q)}$. Since $\widetilde{G}(z)$ is the Laplace transform of a Dirichlet series bounded on the real axis, by Lemma 2 it can be represented as the sum of two functions

$$\widetilde{G}(z) = \widetilde{G}_1(z) + \sum_{n=1}^{\infty}\frac{c_n}{\lambda_n - z},$$

where $\widetilde{G}_1(z)$ is regular and bounded in the half-plane $\operatorname{Re} z \geqslant 0$. Since, by Lemma 5,

$$\ln|P(re^{i\varphi})| \leqslant \frac{\pi r^q}{c^q}|\sin q\varphi|\,(1+o(1)) \qquad (r\to\infty),$$

then for $\varphi = 0$

$$\ln|P(x)| \leqslant o(x^q) \qquad (x\to+\infty).$$

Hence,

$$\ln|P(x)\widetilde{G}_1(x)| \leqslant o(x^q) \qquad (x\to+\infty).$$

Now consider the product

$$P(x)\sum_{n=1}^{\infty}\frac{c_n}{\lambda_n - x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^{2q}}{\lambda_n^{2q}}\right)\cdot\sum_{n=1}^{\infty}\frac{c_n}{\lambda_n - x}.$$

Construct around each λ_n a circle of radius n^{-2}; outside the circles we have the estimate

$$\left|\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - x}\right| \leqslant \sum_{n=1}^{\infty} |c_n| n^2.$$

Recall now that

$$c_n = a_n \int_0^{\infty} e^{-\alpha_1 t^p + \lambda_n t}\, dt.$$

Estimating a_n through $M_F(x)$, and the integral as in Lemma 4, it is easy to see that, since $\alpha_1 > \alpha - \varepsilon$, the inequality

$$|c_n| < Ce^{-\varepsilon_1 n}, \qquad \varepsilon_1 > 0$$

holds. Therefore, outside the constructed circles,

$$\left|\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - z}\right| \leqslant C \sum_{n=1}^{\infty} n^2 e^{-\varepsilon_1 n} = C_1.$$

Since inside the circles our function is regular, and the sum of the radii of the circles forming a connected set does not exceed two, by the maximum modulus principle inside the circles we have

$$\left|P(x) \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - x}\right| \leqslant o\big((x-2)^q\big) \leqslant o(x^q) \qquad (x \to +\infty).$$

Thus, on the real axis $|P(x)\widetilde{G}(x)| \leqslant e^{o(x^q)}$. Now consider the indicator $H_\chi(\varphi)$ of the function $\chi(z)$ in the angle $0 \leqslant \varphi \leqslant \frac{\pi}{2}$. From formula (5)

and from the preceding argument it follows that

$$H_\chi(0) \leqslant -\varliminf_{x \to +\infty} \frac{\ln |E(x)|}{x^q}.$$

According to the estimate of Lemma 4, from this we obtain

$$H_\chi(0) \leqslant -(p-1)\alpha_1^{-\frac{1}{p-1}} p^{-q} = -c_1.$$

On the other hand, from formula (4), taking into account that $|F(iy)| \leqslant M$, we find

$$H_\chi\left(\frac{\pi}{2}\right) \leqslant \lim_{y\to+\infty} \frac{\ln|P(iy)|}{y^q} = \pi c^{-q} \sin\frac{\pi q}{2} = c_2 \sin\frac{\pi q}{2}.$$

By the theorem on the trigonometric q-convexity of the indicator, we construct the function $K(\varphi) = a_1 \cos q\varphi + a_2 \sin q\varphi$ with the conditions $K(0) = H_\chi(0)$ and $K(\pi/2) = H_\chi(\pi/2)$. Then, since $q < 2$, the inequality $H_\chi(\varphi) \leqslant K(\varphi)$ will be valid. We have $a_1 = -c_1$ and $a_2 = K(\pi/2q)$. Here a_2 is determined from the equation

$$-c_1 \cos\frac{\pi q}{2} + a_2 \sin\frac{\pi q}{2} = c_2 \sin\frac{\pi q}{2},$$

or, recalling the values of c_1 and c_2,

$$a_2 = \pi c^{-q} + (p-1)p^{-q} a_1^{-\frac{1}{p-1}} \operatorname{ctg}\frac{\pi q}{2}.$$

Taking into account the definition of the constant α,

$$\pi c^{-q} + (p-1)p^{-q} \alpha^{-\frac{1}{p-1}} \operatorname{ctg}\frac{\pi q}{2} = 0$$

and the relation $a_1 < \alpha$, we obtain that $a_2 < 0$, i.e. $K(\pi/2q) < 0$. Hence, $H_\chi(\pi/2q) < 0$. By symmetry also $H_\chi(-\pi/2q) < 0$.

If we now consider the function $\Phi(z) = \chi(z^q)$, then from the last estimates we obtain that $\Phi(z)$ is regular in the half-plane $\operatorname{Re} z \geqslant 0$ and satisfies the following conditions: a) on the imaginary axis $|\Phi(z)| \geqslant Ce^{-\varepsilon_2|y|}$; b) in the right half-plane $|\Phi(z)| = e^{o(z)}$.

According to a uniqueness theorem for analytic functions (see [(3)]), it follows from this that $\Phi(z) \equiv 0$. Hence successively we find $\chi(z) \equiv 0$, $\widetilde{F}(z) \equiv 0$, and, finally, $F(z) \equiv 0$. Theorem 1 is proved.

We shall also give, without proof, a result establishing that the constant α found is exact for $1 < q < 2$.

Theorem 2. Let λ_n be an increasing sequence of positive numbers, with $\lambda_n \sim cn^{1/q}$, $0 < c < \infty$, $1 < q < 2$. Suppose, moreover, that this sequence satisfies the condition

$$|\lambda_n^q - \lambda_k^q| \geqslant h|n-k|, \qquad h > 0.$$

Then the function

$$Q(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \prod_{n=1}^{\infty} \frac{\lambda_n + \xi}{\lambda_n - \xi} e^{-2\frac{\xi}{\lambda_n}} e^{z\xi} \frac{d\xi}{(\xi+1)^2}$$

is a Dirichlet series of the form (1), not identically zero, bounded on the real axis, and satisfying the condition

$$M_Q(x) \leqslant C_\varepsilon e^{(\alpha+\varepsilon)x^p}, \qquad \varepsilon > 0 \text{ arbitrary.}$$

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Note: Figure translations are in progress. See original paper for figures.