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Fig. 1

Figure 1: Fig. 1

Abstract

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INTEGRAL REPRESENTATION OF THE VERTEX PART IN PERTURBATION THEORY

(Presented by Academician N. N. Bogoliubov, August 14, 1962)

1. In papers ([1],[2]) it was shown that the diagram D, shown in Fig. 1, majorizes all strongly connected diagrams of the nucleon-photon vertex part* (the dashed lines in Fig. 1 are π-meson lines, with mass m; the solid line is a nucleon line, with mass M; all vertices are external). This means that in the space of Euclidean external momenta, in the region G_E in which the quadratic form of the diagram D is less than zero,

$$Q_D(\alpha, p^2) = \frac{\alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_3 p_2^2 + \alpha_1\alpha_2 p_3^2}{\alpha_1 + \alpha_2 + \alpha_3} - (\alpha_1 + \alpha_2)m^2 - \alpha_3 M^2 < 0 \qquad (1)$$

for all $\alpha_\nu \geqslant 0$, not simultaneously equal to zero, the quadratic form Q of any strongly connected diagram of the vertex part is negative.

Fig. 1

This result will be used in the present note for two purposes: 1) to find a certain region in the space of the complex variables

$$z_1 = p_1^2, \qquad z_2 = p_2^2, \qquad z_3 = p_3^2, \qquad (2)$$

in which the contribution of any strongly connected diagram of the vertex part is analytic; 2) to derive an integral representation for an arbitrary diagram which completely reflects the established analytic properties of the vertex part**. From these results there follows a spectral representation of the vertex part in two variables (of the type proposed in ([3])), as well as the usual one-dimensional dispersion relations in any order of perturbation theory.

In Sec. 4 an integral representation is given for the nucleon-nucleon scattering amplitude, the derivation of which is analogous to the derivation of the representation of the vertex part.

2. We shall prove the analyticity of the vertex part in the complex region $\widetilde{G}$ in three stages. First we define the Euclidean region G_E in which the form $Q(\alpha, z)$ of any diagram of the vertex part is negative; then we find a region $G \supset G_E$, consisting of all real z ([2]) for which $Q(\alpha, z) < 0$, and, finally, we define the complex neighborhood $\widetilde{G}$ of the region G, in which the contribution of an arbitrary diagram is analytic.

The Euclidean region E in the space (z_1, z_2, z_3) consists of real points z satisfying the inequalities

$$E: \qquad z_i \geqslant 0, \qquad z_1^2 + z_2^2 + z_3^2 \leqslant 2(z_1 z_2 + z_1 z_3 + z_2 z_3) \tag{3}$$

(when the inequalities (3) are fulfilled, the linear hull of the vectors p, connected –

* As usual, the class of diagrams considered is that whose internal lines correspond to strongly interacting particles. The electromagnetic interaction is taken into account only in first order of perturbation theory (vertex 3 is the only electromagnetic vertex of diagram D). Everywhere below we shall speak of strongly connected diagrams which do not fall apart into two parts after cutting any one internal line.

** A representation of this type was proposed by Nakanishi ([4]).

obtained from z by means of (2), is a Euclidean space). In the region E, according to the result of papers ([1,2]) formulated above, it is sufficient to investigate the third-order diagram D. This diagram has been well studied ([5]). The region G_D in which the form (1) is negative is most simply written in the variables ξ (to preserve the symmetry of the notation, we denote the mass of line ν of the diagram D by m_ν; in the case under consideration $m_1 = m_2 = m,\ m_3 = M$):

$$\xi_1 = \frac{m_2^2 + m_3^2 - z_1}{2m_2 m_3}, \qquad \xi_2 = \frac{m_1^2 + m_3^2 - z_2}{2m_1 m_3}, \qquad \xi_3 = \frac{m_1^2 + m_2^2 - z_3}{2m_1 m_2}. \tag{4}$$

In the variables ξ, the region G_D is specified by the inequalities ([5])

$$G_D \begin{cases} \xi_i > -1 \quad (\text{i.e. } z_i < (m_j + m_k)^2), \quad i = 1, 2, 3; \\ \text{if } \xi_i + \xi_j < 0, \text{ then } \xi_k > \xi_i \xi_j - \sqrt{(1-\xi_i^2)(1-\xi_j^2)}, \end{cases} \tag{5}$$

where (i, j, k) is any permutation of the numbers $(1, 2, 3)$.

The Euclidean region of analyticity is determined by the equality $G_E = G_D \cap E$. For what follows it is essential that (as is not hard to verify) the boundary of the region G_E contains the entire "curvilinear" part of the boundary of the region G_D, i.e. that part of the surface

$$\xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2\xi_3 = 1, \tag{6}$$

which is given by the parametric equations

$$\xi_l = \cos\theta_l, \quad l = 1, 2, 3; \qquad \theta_k = \theta_i + \theta_j \quad \text{for } \theta_i + \theta_j > \pi \quad (0 < \theta_l < \pi). \tag{7}$$

To find the maximal real region of analyticity G, we note that the quadratic form Q of any vertex-part diagram can be written in the form

$$Q(\alpha, z) = A_1(\alpha)z_1 + A_2(\alpha)z_2 + A_3(\alpha)z_3 - \sum_{\nu=1}^{l} \alpha_\nu m_\nu^2, \tag{8}$$

where $A_i(\alpha) \geq 0$ are homogeneous functions of first degree with respect to α (see, for example, ([4]), § 6, IV). From the nonnegativity of the coefficients $A_i(\alpha)$ it follows that if $Q(\alpha, z^0) < 0$ for some z^0, then $Q(\alpha, z) < 0$ for all $z \leq z^0$ (i.e. $z_i \leq z_i^0,\ i = 1, 2, 3$). Thus we are convinced that the region $G = G_D$.

Finally, a complex region of analyticity $\widetilde{G} \supset G$ can be found by means of the device used by Vu ([6]) in the study of the scattering amplitude of scalar particles with equal mass. From the linearity of the form Q (8) with respect to z it follows that $Q(\alpha, z) \neq 0$ (for all nonnegative α, not all simultaneously zero) at the complex point $z = x + iy = (x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$, if there exists a real number λ such that the point $x + \lambda y \in G$. We define $\widetilde{G}$ as the set of complex vectors z for which there exists a λ with the properties just listed.*

3. Let each line of the diagram correspond to the scalar propagation function** $(k_\nu^2 - m_\nu^2 + i0)^{-1}$, and let $2n > l+2$, where l is the number of internal lines and n is the number of vertices of the diagram. The contribution of any strongly connected vertex-part diagram of this type is proportional to a func-

* If complex vectors $z = x + iy$ are represented as applied vectors in a three-dimensional real space, with origin at the point x and endpoint at the point $x+y$, and to each such vector there is assigned the straight line passing through it, then the region $\widetilde{G}$ can be characterized geometrically as the set of vectors to which there correspond straight lines intersecting G (J. Bros—private communication).

** All the arguments and results of this section are easily generalized to the case where certain lines of the diagram correspond to spinor or vector particles. One of the possible methods of such a generalization is described in ([4]) (§§ 3 and 4).

tion (see (4,7))

$$F(z) = \int_0^1 \cdots \int_0^1 D^{-2}(\alpha) \frac{\delta\left(1 - \sum_{\nu=1}^l \alpha_\nu\right) \prod_{\nu=1}^l d\alpha_\nu}{[Q(\alpha, z) + iQ]^{2n-l-2}}, \tag{9}$$

where $D(\alpha)$ is a homogeneous function of degree $l - n + 1$ in α, positive for $\alpha_\nu > 0$, and Q is given by an expression of type (8). From (9), by the change of variables (4)

$$\eta_i = [A_1(\alpha) + A_2(\alpha) + A_3(\alpha)]^{-1} A_i(\alpha), \qquad i = 1, 2, 3;$$

$$\rho = \left[\sum_{i=1}^3 A_i(\alpha)\right]^{-1} \sum_{\nu=1}^l \alpha_\nu m_\nu^2 \tag{10}$$

and by $(2l-n-3)$-fold integration by parts with respect to ρ, we obtain the following integral representation for any strongly connected vertex-part diagram:

$$F(z) = \int_0^1 \int_0^1 \int_0^1 d\eta_1\, d\eta_2\, d\eta_3 \int_{\rho_0(\eta)}^\infty d\rho\, \frac{f(\eta, \rho)}{\eta z - \rho + i\theta}\, \delta(1 - \eta_1 - \eta_2 - \eta_3), \tag{11}$$

where

$$\rho_0(\eta) = \max_{z \in G} \eta z, \qquad \eta z = \eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3, \tag{12}$$

and $f(\eta, \rho)$ is a certain generalized function (the integral (11) should be understood as the limit, as $\varepsilon \to +0$, of integrals over the whole space of variables η and ρ of the same integrand multiplied by an infinitely smooth function $\varphi_\varepsilon(\eta, \rho)$, equal to 1 in the domain of integration (11) and vanishing outside an ε-neighborhood of this domain). The maximum (12) can be calculated explicitly if one observes that it is attained on a part of the boundary of the domain G_E, specified by the parametric equations (7). It is equal to

$$\rho_0(\eta) = \Phi(\eta) \sum_{i=1}^3 \frac{m_i^2}{\eta_i} \equiv \rho(\eta), \qquad \text{if } \left|\frac{m_1}{\eta_1} - \frac{m_2}{\eta_2}\right| \le \frac{m_3}{\eta_3} \le \frac{m_1}{\eta_1} + \frac{m_2}{\eta_2}; \quad (13\ )$$

$$\rho_0(\eta) = \eta_1(m_2 + m_3)^2 + \eta_2(m_1 + m_3)^2 + \eta_3(m_1 - m_2)^2, \qquad \text{if } \frac{m_3}{\eta_3} > \frac{m_1}{\eta_1} + \frac{m_2}{\eta_2}; \quad (13\ )$$

$$\rho_0(\eta) = \eta_1(m_2+m_3)^2 + \eta_2(m_1-m_3)^2 + \eta_3(m_1+m_2)^2, \qquad \text{if } \frac{m_2}{\eta_2} > \frac{m_1}{\eta_1} + \frac{m_3}{\eta_3}; \tag{13 }$$

$$\rho_0(\eta) = \eta_1(m_2-m_3)^2 + \eta_2(m_1+m_3)^2 + \eta_3(m_1+m_2)^2, \qquad \text{if } \frac{m_3}{\eta_3} > \frac{m_1}{\eta_1} + \frac{m_2}{\eta_2}; \tag{13 }$$

here (as also in Sec. 2) $m_1 = m_2 = m,\ m_3 = M$, and the function $\Phi(\eta)$ is equal to

$$\Phi(\eta) = \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3. \tag{14}$$

In the special case of the diagram D (Fig. 1), it is not difficult to find the explicit form of the weight function $f(\eta,\rho)$ in the representation (11): $f_D(\eta,\rho) = \Phi^{-1}(\eta)\delta(\rho-\rho(\eta))$ ($\rho(\eta)$ is specified by formula (13)).

The representation (11), with (13) taken into account, is a certain refinement of the representation found by Nakanishi (4) (in (4) the exact value of $\rho_0(\eta)$ is not determined). The kernel of this representation, $(\eta z-\rho)^{-1}$, gives an example of a three-parameter family of functions analytic in $\widehat{G}$, with parameters ρ and η from the domain of integration in (11), such that for every point of the boundary $\partial\widehat{G}$ of the domain $\widehat{G}$ one can find a certain function of this family,

having a singularity at this point.* Hence, in particular, it follows that the domain $\widetilde{G}$ is the natural domain of holomorphy in the space of the three complex variables (2). From (11), by the change of variables

$$\eta_1 = \lambda\xi, \qquad \eta_2 = \lambda(1-\xi), \qquad \rho = \lambda\gamma + (1-\lambda)z_3 \tag{15}$$

and by integration with respect to λ and η_3, we obtain the integral representation of the vertex part as a function of two variables z_1 and z_2 (with z_3 fixed), derived by Deser et al. [3] on the basis of an incorrect Dyson representation for the double commutator (see also [4]).

The usual one-dimensional dispersion relations can be obtained both from the integral representation (11) and (more simply) directly, by applying Cauchy' s theorem, starting from the domain of analyticity $\widetilde{G}$ obtained in Sec. 2.

4. Starting from the results on majorization of scattering diagrams [8], one can derive, analogously to Sec. 3, integral representations for scattering amplitudes. We shall give the result of such an investigation for the case of nucleon-nucleon scattering. In this case the domain G_{NN}, in which $Q<0$ for all strongly connected diagrams of this process, is the triangle

$$s=(p_1+p_2)^2<4M^2,\qquad t=(p_1+p_3)^2<4m^2,\qquad u=(p_2+p_3)^2<4m^2 \tag{16}$$

in the plane $s+t+u=4M^2$ (we take all external lines to be incoming). The contribution of strongly connected diagrams to the nucleon-nucleon scattering amplitude can be represented in the form**

$$T(s,t)=\int_0^1 d\alpha\left\{\int_{4m^2}^{\infty}d\rho\,\frac{f_1(\alpha,\rho)}{\alpha t+(1-\alpha)u-\rho}+\int_{\rho_0(\alpha)}^{\infty}d\rho\,\frac{f_2(\alpha,\rho)}{\alpha s+(1-\alpha)u-\rho}+\right.$$

$$\left.+\frac{f_3(\alpha,\rho)}{\alpha s+(1-\alpha)t-\rho}\right\}, \tag{17}$$

where

$$\rho_0(\alpha)=4[M^2\alpha+m^2(1-\alpha)]. \tag{18}$$

To obtain (17), it is necessary to divide the region of integration with respect to α in the Feynman integral into parts, in each of which, among the three (dependent) variables s, t, and u, there is a pair such that the coefficients of these variables in the form $Q(\alpha,\rho)$ are nonnegative in the given part of the integration region.***

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* If $z = x + iy \in \partial\widetilde{G}$ and $y \neq 0$, then it is necessary to choose η so that $\eta y = 0$ and $\eta x = \rho(\eta)$.

** A representation of this type was first obtained (by another method) by Nakanishi [(9)].

*** The possibility of such a partition of the integration region was noted by D. Ya. Petrina (private communication).

Note: Figure translations are in progress. See original paper for figures.