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Abstract

Full Text

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LEFT IDEALS OF THE SEMIGROUP OF ENDOMORPHISMS OF A FREE UNIVERSAL ALGEBRA

(Presented by Academician P. S. Novikov on 6 XII 1962)

Let G be a free algebra in some primitive class of universal algebras Λ . In the present paper a description is given of the structures of all left and all two-sided ideals of the semigroup of endomorphisms of the algebra G , by means of subalgebras and certain other subsets of the algebra G itself. The results obtained can naturally be applied to various concrete examples. We note only that, if the set of operations is empty, then from the results of § 5 one easily obtains the description, given in the work of A. I. Mal'cev⁽²⁾, of the two-sided ideals of the semigroup of all mappings of a certain set into itself. Basic information on the theory of universal algebras and semigroups can be found in the books^(1,3).

The set of all subalgebras of a universal algebra, generally speaking, is not a lattice, because the intersection of subalgebras may turn out to be empty. However, this set becomes a complete lattice if it is supplemented by the empty set. The same situation may occur also for certain other kinds of subalgebras or subsets of a universal algebra (for example, for left and two-sided ideals of some semigroup). In what follows we shall assume that, whenever it is necessary, such a supplementation has been made.

Let x_i , where i ranges over an index set I of cardinality n , be some system of free generators of the algebra G . Denote by $A(G)$ the semigroup of all endomorphisms of the algebra G .

1. We shall call a subalgebra F of the algebra G an n -subalgebra if it has at least one system of generators of cardinality not exceeding n .

Denote by Q the set of all collections of n -subalgebras of the algebra G that have the following property:

- a) If the n -subalgebra F belongs to the collection q , and H is an n -subalgebra such that $H \subset F$, then H also belongs to the collection q .

The set Q is a complete lattice with respect to the set-theoretic operations of union and intersection.

If M is some union of subalgebras of the algebra G , then by \mathfrak{A}_M we denote the left ideal consisting of all endomorphisms that map the algebra G into M . Left

ideals of the form \mathfrak{A}_M will be called saturated left ideals of the semigroup $A(G)$.

The complete lattice Σ of all left ideals of the semigroup $A(G)$ is isomorphic to the complete lattice Q .

There exists an isomorphism between the lattices Σ and Q such that, if under this isomorphism the collection $q \in Q$ corresponds to the left ideal $\mathfrak{A} \in \Sigma$, then

$$\mathfrak{A} = \bigcup_{F \in q} \mathfrak{A}_F.$$

2. Denote by U the set of all unions of certain subalgebras of the algebra G . It is easy to see that U is a complete lattice with respect to the set-theoretic operations of intersection and union.

The set Π of all saturated left ideals of the semigroup $A(G)$ is a complete lattice isomorphic to the complete lattice U .

The lattice Π is a sublattice of the lattice Σ if and only if the algebra G is cyclic (i.e., an algebra with one generator); moreover, in this case Π and Σ coincide.

3. In what follows, by M we shall denote only **unions of certain subalgebras** of the algebra G , i.e. some element of the lattice U . For each M , denote by \mathfrak{A}_M the set of all endomor-

morphisms from $A(G)$ mapping G into cyclic subalgebras of M . We shall say that the left ideal $\mathfrak{A} \in \Sigma$ maps G onto M , and shall write $M_{\mathfrak{A}} = M$, if

$$M = \bigcup_{\alpha \in \mathfrak{A}} G\alpha.$$

$\overline{\mathfrak{A}}_M$ is a left ideal of the semigroup $A(G)$, and the least among all those left ideals which map G onto M .

The set π of all left ideals of the form $\overline{\mathfrak{A}}_M$ is a complete lattice, isomorphic to the complete lattice U , and consequently also to the complete lattice of all saturated left ideals Π .

The lattice π is a sublattice of the lattice Σ if and only if, for every n -subalgebra contained in some cyclic subalgebra, there exists a unique least cyclic subalgebra containing the given n -subalgebra.

For each $M \in U$, denote by Σ_M the set of all left ideals $\mathfrak{A} \in \Sigma$ which map G onto M , i.e., for which $M_{\mathfrak{A}} = M$. Let σ be the partition of the lattice Σ into the classes Σ_M .

The partition σ is a congruence of the complete lattice Σ .

The quotient lattice Σ/σ is isomorphic to the complete lattice U .

4. Each class Σ_M of the congruence σ is a complete lattice whose zero and unit are, respectively, the left ideals $\overline{\mathfrak{A}}_M$ and \mathfrak{A}_M .

Denote by Q_M the set of all families of n -subalgebras possessing property a) of item 1 and the property:

- b) the union of all n -subalgebras of any family in Q_M is equal to M .

The set Q_M is a complete lattice with respect to the set-theoretic operations of union and intersection.

It follows from item 1 that the complete lattices Σ_M and Q_M are isomorphic.

5. A family of subsets $N_i \subseteq G$ is called fully characteristic if, for every subset N_i of this family and every endomorphism $\alpha \in A(G)$, there exists a subset $N_{i'}$ of the same family such that $N_i \alpha \subseteq N_{i'}$. If $N_i = N$ for all i , then the set N itself is fully characteristic.

Denote by \mathfrak{N} the set of all fully characteristic families of n -subalgebras $q \in Q$. The set \mathfrak{N} is a complete lattice with respect to the set-theoretic operations of intersection and union.

The complete lattice Δ of all two-sided ideals of the semigroup $A(G)$ is isomorphic to the complete lattice \mathfrak{N} .

The left ideals \mathfrak{A}_M and $\overline{\mathfrak{A}}_M$ are two-sided ideals of the semigroup $A(G)$ if and only if the subset $M \in U$ is fully characteristic.

It follows from this that the two-sided ideals of the semigroup $A(G)$ are distributed among the classes Σ_M for fully characteristic M , and each such class contains two-sided ideals; namely, in every case the ideals $\overline{\mathfrak{A}}_M$ and \mathfrak{A}_M are such.

6. A partial endomorphism of the algebra G is any homomorphism α of some subalgebra Γ_α^1 of the algebra G onto a subalgebra Γ_α^2 of G . Multiplication of partial endomorphisms is defined in the same way as multiplication of partial transformations of some set ⁽³⁾. Denote by $W(G)$ the semigroup of all partial endomorphisms of the algebra G .

As in item 1, denote by \mathfrak{A}_M^W , $M \in U$, the set of all partial endomorphisms $\alpha \in W(G)$ for which $\Gamma_\alpha^2 \subseteq M$. \mathfrak{A}_M^W is a left ideal of the semigroup $W(G)$. The left ideals \mathfrak{A}_M^W are called saturated left ideals of the semigroup $W(G)$.

The set Π^W of all saturated left ideals of the semigroup $W(G)$ is a complete lattice, isomorphic to the complete lattice of all unions of subalgebras of the algebra G .

7. We shall say that a **left ideal** \mathfrak{A} of the semigroup $W(G)$ **partially maps** G onto M , and shall write $M_{\mathfrak{A}} = M$, if

$$M = \bigcup_{\alpha \in \mathfrak{A}} \Gamma_\alpha^2.$$

Let Σ^W be the structure of all left ideals of the semigroup $W(G)$. Denote by Σ_M^W the set of all left ideals $\mathfrak{A} \in \Sigma^W$ for which $M_{\mathfrak{A}} = M$. If σ is the partition of the structure Σ^W into the classes Σ_M^W , then:

The partition σ is a congruence of the complete structure Σ^W . The factor-structure Σ^W/σ is isomorphic to the complete structure U of all unions of subalgebras of the algebra G .

It is obvious that each class Σ_M^W contains a unique maximal left ideal, namely \mathfrak{A}_M^W . Each such class also contains a unique minimal left ideal $\overline{\mathfrak{A}}_M^W$. The set π^W of all left ideals $\overline{\mathfrak{A}}_M^W$ is also a complete structure, isomorphic to the complete structure U . Always

$$\overline{\mathfrak{A}}_M^W \supset \mathfrak{A}_M^W,$$

and, generally speaking, the inclusion is proper.

8. In what follows only those free algebras G are considered for which the theorem on the freeness of subalgebras holds, i.e. every subalgebra of the free algebra G is itself a free algebra in the same primitive class Λ . Denote by Q' the set of all collections of subalgebras of the algebra G possessing the property:

a') together with some subalgebra F , the given collection contains all subalgebras contained in F .

The set Q' is a complete structure with respect to the set-theoretic operations of union and intersection.

The complete structures Σ^W and Q' are isomorphic.

Let us note that if the set I is infinite and its cardinality n is not less than the cardinality of the set of fundamental operations of the primitive class Λ , which, as usual, are assumed to be finitary, then every subalgebra of the algebra G is an n -subalgebra. Consequently, in this case the sets Q and Q' coincide, whence we obtain that the complete structures of all left ideals of the semigroup $A(G)$ and of all left ideals of the semigroup $W(G)$ are isomorphic. In particular, these structures are isomorphic if G is a free group with a countable number of generators.

9. We shall call a cardinal number m **admissible** if: 1) for every $m' < m$ one can indicate such an m'' , $m' \leq m'' < m$, that the algebra G contains a subalgebra which has a system of generators of cardinality m'' , but has no system of generators of smaller cardinality; 2) if a subalgebra $H \subseteq G$ has a system of generators of cardinality less than m , then every subalgebra of the algebra H also has a system of generators whose cardinality is less than m .

For example, if G is a noncyclic free group of finite or countable rank, then only the numbers 1, 2, and \aleph_1 are admissible. It follows from the definition that an admissible cardinal number m does not exceed, in any case, $\aleph_{\nu+1}$, where \aleph_ν is the cardinality of the algebra G .

Denote by \mathfrak{A}_m the set of all partial endomorphisms $\alpha \in W(G)$ for which Γ_α^2 has a system of generators whose cardinality is strictly less than the admissible

cardinal number m . Then:

\mathfrak{A}_m is a two-sided ideal of the semigroup $W(G)$. If m_1 and m_2 are two admissible cardinal numbers and $m_1 < m_2$, then $\mathfrak{A}_{m_1} \subset \mathfrak{A}_{m_2}$. Every two-sided ideal \mathfrak{A} of the semigroup $W(G)$ coincides with one of the ideals \mathfrak{A}_m .

I take this opportunity to express my deep gratitude to Prof. A. G. Kurosh for his constant attention to the work and valuable advice.

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CITED LITERATURE

¹ A. G. Kurosh, *Lectures on General Algebra*, Moscow, 1962.

² A. I. Maltsev, *Matem. sborn.*, **31** (73), 136 (1952).

³ E. S. Lyapin, *Semigroups*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.

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