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**Abstract**

**Full Text**

**S. D. EIDELMAN**

**ON THE THEORY OF GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC SYSTEMS**

*(Presented by Academician I. N. Vekua on 17 IX 1962)*

The note consists of two parts. In the first part a solution is given of the general boundary-value problem for a half-space under substantially more general assumptions on the solvability condition than in (1).

The general boundary-value problem in a half-space was studied by T. Ya. Zagorskii (4). The results obtained by him differ from ours.

The second part of the note is devoted to the exposition of the results of an investigation of boundary-value problems for systems with variable coefficients and boundary conditions in the case of one spatial coordinate.

1. We consider the problem of finding a solution  $u(t, x)$  of the parabolic system

$$\frac{du}{dt} = \sum_{|k|=2b} A_k D^k u \equiv A(D)u \tag{1}$$

in the domain  $\Pi\{0 < t \leq T, -\infty < x_s < \infty, s = 1, 2, \dots, n-1, x_n > 0\}$ , satisfying the conditions

$$\begin{aligned} u(+0, x) = 0, \quad x = (x_1, x_2, \dots, x_n), \quad -\infty < x_s < \infty, \\ s = 1, 2, \dots, n-1, \quad x_n > 0; \end{aligned} \tag{2}$$

$$\begin{aligned} \lim_{x_n \rightarrow +0} B \left( \frac{\partial}{\partial t}, D \right) u &= \lim_{x_n \rightarrow +0} \left( \sum_{j=1}^N \sum_{2bk_0 + |k|=r_i} b_{ij}^{(k_0, k)} \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j \right)_{i=1, 2, \dots, bN} \\ &= f(t, x'); \end{aligned} \tag{3}$$

$$0 < t \leq T, \quad x' = (x_1, x_2, \dots, x_{n-1}), \quad -\infty < x_s < \infty, \quad s = 1, 2, \dots, n-1,$$

$$D^k = (-i\partial/\partial x_1)^{k_1} \dots (-i\partial/\partial x_n)^{k_n}, \quad k = (k_1, k_2, \dots, k_n).$$

Consider the problem of finding, for  $x_n > 0$ , a solution  $v(x_n; p, \sigma')$  of the system of ordinary differential equations

$$A\left(p, \sigma'; -i\frac{d}{dx_n}\right)v \equiv \left(A\left(\sigma', -i\frac{d}{dx_n}\right) - pE\right)v = 0, \quad (4)$$

satisfying the boundary conditions

$$B\left(p, \sigma'; -i\frac{d}{dx_n}\right)v \Big|_{x_n=0} = g(p, \sigma'); \quad (5)$$

$$g(p, \sigma') = \frac{1}{(2\pi)^{n_i}} \int_0^\infty e^{-p\tau} d\tau \int e^{-i(\sigma', \xi')} f(\tau, \xi') d\xi'$$

and belonging to the space  $L_2(0, \infty)$ ;  $\sigma' = (\sigma_1, \dots, \sigma_{n-1})$  is an arbitrary real vector,  $p = a_0 + ip_1$ ,  $a_0$  is a certain positive constant,  $-\infty < p_1 < \infty$ . Denote by  $H_r$ ,  $r = bN$ , the  $r$ -dimensional subspace of solutions of system (4) belonging to  $L_2(0, \infty)$ , and by  $\{v_i(x_n; p, \sigma')\}_{i=1,2,\dots,r}$  a basis of this subspace; then the necessary and sufficient condition for unique solvability in  $H_r$  of problem (4), (5) for arbitrary  $g(p, \sigma')$  consists in the fact that, for all indicated values of the parameters  $p, \sigma'$ ,

$$\det B\left(p, \sigma'; -i\frac{d}{dx_n}\right)V(x_n; p, \sigma') \Big|_{x_n=0} \neq 0; \quad (6)$$

$$V(x_n; p, \sigma') = (v_1(x_n; p, \sigma'), v_2(x_n; p, \sigma'), \dots, v_r(x_n; p, \sigma')).$$

Inequality (6) is precisely the solvability condition for problem (1)–(3) (condition P)

Any vector from  $H_r$  can be obtained from the formula

$$v(x_n; p, \sigma') = \int_{\Gamma^+} e^{ix_n\sigma_n} A^{-1}(p, \sigma) M(\sigma_n) d\sigma_n C; \quad (7)$$

$$M(\sigma_n) = (E, \sigma_n E, \dots, \sigma_n^{2b-1} E);$$

$\Gamma^+$  is a contour in the upper half-plane of the complex  $\sigma_n$ -plane, enclosing all  $\sigma_n$ -roots of the equation  $\det A(p, \sigma) = 0$  with  $\text{Im } \sigma_n > 0$ , with a suitable choice of the vector  $C$  (with  $2r$  coordinates).

If (7) is used, then condition (6) can be given the form <sup>(4,6)</sup>

$$\text{rank} \int_{\Gamma^+} B(p, \sigma) A^{-1}(p, \sigma) M(\sigma_n) d\sigma_n = r. \quad (8)$$

With the aid of formula (7), the solution of problem (4), (5) is written in a form not convenient for further investigations. Below, convenient formulas will be obtained for the solution of problem (4), (5); in doing so, the fulfillment of condition P in the case of the first boundary-value problem (3) is used. The arguments remain valid for elliptic systems for which condition P is fulfilled (see (6), (8),  $p = 0$ ,  $\sigma' \neq 0$ ) in the case of the first boundary-value problem.

**Lemma 1\*.** *If condition P is fulfilled in the case of the first boundary-value problem (3), then*

$$F_1(p, \sigma') = \det \int_{\Gamma^+} M_1'(\sigma_n) A^{-1}(p, \sigma) M_1(\sigma_n) d\sigma_n \neq 0, \quad (9)$$

$$M_1(\sigma_n) = (E, \sigma_n E, \dots, \sigma_n^{b-1} E)$$

for real  $\sigma'$  and  $p$  from the half-plane  $\text{Re } p \geq 0$ ,  $(p, \sigma') \neq 0$ .

**Lemma 2 (on factorization) (2).** *If inequality (9) is fulfilled, then there exists a matrix*

$$A_+(p, \sigma) = \sum_{j=0}^b A_{b-j}^+(p, \sigma') \sigma_n^j, \quad A_0^+(p, \sigma') = E,$$

whose  $\sigma_n$ -roots of the equation  $\det A_+(p, \sigma) = 0$  coincide with the roots of the equation  $\det A(p, \sigma) = 0$  with  $\text{Im } \sigma_n > 0$ , which divides the matrix  $A(p, \sigma)$  on the right, i.e.

$$A(p, \sigma) = A_-(p, \sigma) A_+(p, \sigma),$$

$A_-(p, \sigma)$  being a matrix polynomial in  $\sigma_n$ . Moreover,  $A_{b-j}^+(p, \sigma')$ ,  $j = 0, \dots, b$ , are analytic functions of  $(p, \sigma')$  for the values of these arguments indicated in Lemma 1.

Using Lemma 2, the vectors from  $H_r$  can be represented as follows:

$$v(x_n; p, \sigma') = \int_{\Gamma} e^{ix_n \sigma_n} A_+^{-1}(p, \sigma) M_1(\sigma_n) d\sigma_n C. \quad (10)$$

Then condition P takes the form

$$F(p, \sigma') = \det \int_{\Gamma} B(p, \sigma) A_+^{-1}(p, \sigma) M_1(\sigma_n) d\sigma_n \neq 0; \quad (11)$$

$p = a_0 + ip_1$ ,  $a_0$  is some positive constant,  $-\infty < p_1 < \infty$ ,  
 $\sigma'$  is an arbitrary real vector.

The solution of problem (4), (5) is determined by the formulas

$$W(x_n; p, \sigma') = \int_{\Gamma} e^{ix_n \sigma_n} Q(p, \sigma) d\sigma_n;$$

$$Q(p, \sigma) = A_+^{-1}(p, \sigma) M_1(\sigma_n) \left[ \int_{\Gamma} B(p, \sigma) A_+^{-1}(p, \sigma) M_1(\sigma_n) d\sigma_n \right]^{-1},$$

and the desired F.M.S. is

$$G(t, x) = \frac{1}{(2\pi)^n i} \int e^{i(x', \sigma')} d\sigma' \int_{a_0 - i\infty}^{a_0 + i\infty} e^{pt} dp \int_{\Gamma} e^{ix_n \sigma_n} Q(p, \sigma) d\sigma_n. \quad (12)$$

\* For the case of elliptic systems with two independent variables, Lemma 1 was proved by A. I. Volpert <sup>(7)</sup>.

Using the study, described in <sup>(1)</sup>, of the domain of analyticity  $Q(p, \sigma)$ , the generalized homogeneity of its columns, and, on this basis, replacing the contour of integration with respect to  $p$  in the complex  $p$ -plane in the proper way, one can prove that, for the p.f.m.s.  $G(t, x)$  defined by formula (12), Theorem 1 <sup>(1)</sup> is valid, and

$$u(t, x) = \int_0^t d\tau \int G(t - \tau, x - \xi') f(\tau, \xi') d\xi'$$

gives a solution of problem (1)–(3), if  $f(\tau, \xi')$  satisfies the conditions stated in Theorem 2 <sup>(2)</sup>.

2. Consider, in the cylinder  $\mathcal{G}_1 = V_1 \times [0, T]$ , the parabolic system

$$L(u) \equiv \frac{\partial u}{\partial t} - \sum_{|k|=2b} A_k(t, x) D^k u - \sum_{|k| \leq 2b-1} A_k(t, x) D^k u \quad (13)$$

$$\equiv \frac{\partial u}{\partial t} - A_0(t, x; D)u - A_1(t, x; D)u = 0.$$

Let  $V, \bar{V} \subset V_1$  be a convex bounded domain with smooth boundary  $S$ ; on the lateral surface  $\Gamma_0$  of the cylinder  $\mathcal{G} = V \times [0, T]$  let a boundary operator be given,

$$\begin{aligned} B(t, x; D) &= \left( \sum_{|k|=r_i} b_{ij}^{(k)}(t, x) D^k + \sum_{|k| \leq r_i - 1} b_{ij}^{(k)}(t, x) D^k \right)_{\substack{i=1,2,\dots,bN \\ j=1,2,\dots,N}} \\ &= B_0(t, x; D) + B_1(t, x; D), \quad r_i \leq 2b - 1. \end{aligned}$$

We state one auxiliary proposition needed for the construction, in the domain  $\mathcal{G}$ , of a solution of system (13) satisfying the conditions

$$u|_{t=+0} = 0; \quad \lim_{x \rightarrow \xi \in S, x \in V} B(t, x; D)u = f(t, \xi). \quad (14)$$

We shall assume that, for problem (13), (14), condition  $P$  is fulfilled, having the form

$$\begin{aligned} \left| \det \int_{\Gamma} B_0(\tau, \xi; \zeta + \mu\nu(\xi)) A_{0+}^{-1}(\tau, \xi; p; \zeta + \mu\nu(\xi)) M_1(\mu) d\mu \right| &\geq \\ &\geq \delta_1 (|\zeta|^2 + |p|^2)^{m/2}; \end{aligned} \quad (15)$$

$$m = \sum_{i=1}^{bN} r_i - \frac{Nb(b-1)}{2};$$

$(\tau, \xi)$  is any point of  $\bar{\Gamma}_0$ ;  $\nu(\xi)$  is the inward normal to  $S$  at the point  $\xi$ ;  $\zeta$  is any real vector orthogonal to  $\nu(\xi)$ ;  $p = a_0 + ip_1$ ;  $a_0$  is a certain positive constant,  $-\infty < p_1 < \infty$ ;  $\Gamma$  is a closed contour in the  $\mu$ -plane enclosing all  $\mu$ -roots of the equation

$$\det A_{0+}(\tau, \xi; p, \zeta + \mu\nu(\xi)) = 0;$$

$\delta_1$  is an absolute positive constant.

Using the results of § 1, one can find the p.f.m.s. of the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= A_0(\tau, \xi; D)u; \quad u|_{t=+0} = 0; \quad \lim_{\substack{x \rightarrow z \\ (x, \nu(\xi)) > 0}} B(\tau, \xi; D)u = f(t, z); \\ (z, \nu(\xi)) &= 0; \quad (\tau, \xi) \in \Gamma_0, \end{aligned} \quad (16)$$

$G^{\nu(\xi)}(t, x; \tau, \xi)$ , for whose columns, when  $(x, \nu(\xi)) > 0$ , the estimates

$$|D^m G_j^{\nu(\xi)}(t, x; \tau, \xi)| \leq C_m t^{-\frac{n-1+2b-r_j+|m|}{2b}} \exp \left\{ -c \left( \frac{|x|}{t^{1/2b}} \right)^q \right\}; \quad q = \frac{2b}{2b-1}. \quad (17)$$

**Definition** (<sup>4 6</sup>). A special solution matrix (s.m.s.) of system (13)  $\mathcal{E}(t, \tau, x, \xi)$ , corresponding to problem (14), is a matrix whose columns are, in  $\mathcal{G}$ , solutions of (13) and which, in a neighborhood of the point  $(\tau, \xi) \in \Gamma_0$ , has as its principal part (with respect to the order of singularity)

$$G^{\nu(\xi)}(t - \tau, x - \xi; \tau, \xi).$$

**Theorem 1.** Let the coefficients of (13) in  $\overline{\mathfrak{B}}$  satisfy the conditions:

1)  $A_k(t, x)$  with  $|k| = 2b$  satisfy the Hölder condition in  $t$  with exponent

$$1 - \frac{r_j + 1}{2b} + \varepsilon;$$

2)  $A_k(t, x)$  have  $|k| - (r_j + 1)$  continuous derivatives with respect to  $x$ , Hölder-continuous in  $x$ ; for  $|k| < r_j + 1$ ,  $A_k(t, x)$  are continuous and Hölder-continuous in  $x$ . Then system (13) has a c.f.s.  $\mathcal{E}(t, \tau, x, \xi)$  corresponding to problem (14), for which the estimates (17) are valid in  $\mathfrak{B}$ ,  $|m| \leq 2b - 1$ .

If the orders of the boundary operators  $r_i$  coincide,  $r_i = r$ ,  $i = 1, 2, \dots, \dots, bN$ , then, by means of the potential

$$v(t, x) = \int_0^t d\tau \int_S \mathcal{E}(t, \tau, x, \xi) \mu(\tau, \xi) d_\xi S,$$

the solution of problem (13), (14) is reduced to the solution of a Volterra integral equation of the second kind with a quasi-regular kernel.

**Theorem 2.** If: 1) the coefficients of (13) satisfy the conditions of Theorem 1; 2) the coefficients of the boundary condition are given in the boundary strip  $\Gamma_0$ , are Hölder-continuous in  $t$  and  $x$ ; 3) the boundary  $S$  belongs to the class  $A^{(1, \alpha)}$  ((9), p. 10); 4)  $f(t, x')$  is a continuous function, then problem (13), (14) is solvable.

In the case  $n = 1$ , the following method of solving problem (14) can be proposed; in this case it has the form

$$u|_{t=+0} = 0; \quad \lim_{x \rightarrow +0} B \left( t, 0, -i \frac{\partial}{\partial x} \right) u(t, x) = \varphi(t),$$

$$\lim_{x \rightarrow 1-0} B \left( t, 1, -i \frac{\partial}{\partial x} \right) u(t, x) = \Psi(t). \quad (18)$$

First one constructs the p.f.m.s.  $G(t - \tau, x - \xi; \tau, \xi)$ ,  $\xi = 0, 1$ , of problem (16) (see (12)); from it the columns of the matrix  $\tilde{G}_j(t - \tau, x - \xi; \tau, \xi)$  are determined by the formulas

$$\tilde{G}_j(t - \tau, x - \xi; \tau, \xi) =$$

$$= \frac{1}{2\pi i} \int_{a_0 - i\infty}^{a_0 + i\infty} p^{-\frac{\bar{r}-r_j}{2b}} e^{p(t-\tau)} dp \int_1 e^{i(x-\xi)\sigma} Q_j(p, \sigma; \tau, \xi) d\sigma;$$

$Q_j$  is from formula (12),  $\bar{r} = \max_j r_j$ , and then a special matrix of solutions of system (13),  $\mathcal{E}(t, \tau, x, \xi)$ , is constructed, whose principal part in a neighborhood of boundary points coincides with  $\tilde{G}(t - \tau, x - \xi; \tau, \xi)$ . The potential whose kernel is  $\mathcal{E}(t, \tau, x, \xi)$  makes it possible to reduce the solution of problem (13), (14) to the solution of a Volterra integral equation of the first kind (if not all  $r_j$  coincide), which, with the aid of fractional differentiation operators <sup>(8, 5)</sup>, is reduced to an equivalent system of Volterra integral equations of the second kind with a quasi-regular kernel.

**Theorem 3.** If the coefficients of system (13) satisfy the conditions of Theorem 1 with  $r_j = \bar{r}$ , and  $\Phi_i(t)$ ,  $\Psi_i(t)$  and the coefficients of the boundary conditions

$$B_i \left( \tau, \xi; -i \frac{\partial}{\partial x} \right), \quad \xi = 0, 1,$$

satisfy the Hölder condition on the segment  $[0, T]$  with exponent

$$\frac{\bar{r} - r_i}{2b} + \varepsilon; \quad \varepsilon > 0,$$

$\varphi_i(0) = \Psi_i(0) = 0$ , if  $r_i < \bar{r}$ , then problem (13), (14) is solvable.

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