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Abstract

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MATHEMATICS

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A REPRESENTATION THEOREM FOR A CLASS OF DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES BY MEANS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

(Presented by Academician I. M. Vinogradov on 25 XII 1962)

In this note we consider a class of functions which I shall denote by $S_p^{(r)}H(R_n)^*$.

Let n be a natural number; $e_n = \{1, \dots, n\}$ is the set of natural numbers (indices) not exceeding n ; $e \subset e_n$ is an arbitrary subset of it. In particular, e may be the empty set ($e = 0$) or may coincide with e_n . Further, let $\mathbf{r} = (r_1, \dots, r_n)$ be a vector (with nonnegative coordinates). Denote by e_r the support of \mathbf{r} , i.e. the set of indices j for which $r_j > 0$. Thus, $r_j > 0$ for $j \in e_r$ and $r_j = 0$ for $j \in e_n - e_r$.

Denote by $\mathbf{r}^e = (r_1^e, \dots, r_n^e)$ a vector such that $r_j^e = r_j$ for $j \in e$ and $r_j^e = 0$ for $j \in e_n - e$. Let also $\omega_n = (1, \dots, 1)$, and let R_n be the n -dimensional real space of points $\bar{x} = (x_1, \dots, x_n)$.

By definition, a function $f(\bar{x})$ belongs to the class $S_p^{(r)}H(R_n) = S_p^{(r)}$ ($1 \leq p \leq \infty$) if the following properties hold:

- 1) $f \in L_p(R_n)$.
- 2) The generalized derivatives $f^{(\mathbf{r}^e)}$ make sense on R_n , where $\mathbf{r} = (r_1, \dots, r_n)$ and $r_j = \bar{r}_j + \alpha_j$, \bar{r}_j is an integer, $0 < \alpha_j \leq 1$.
- 3)

$$\frac{\|\Delta_h^{2\omega_n} f^{(\mathbf{r}^e)}\|_{L_p(R_n)}}{\prod_{j \in e} |h_j|^{\alpha_j}} \leq M_p^{(\mathbf{r}^e)}(f) < \infty \quad (1)$$

for every $e \subset e_r$ and vector $\mathbf{h} = (h_1, \dots, h_n)$ with $h_j > 0$ ($j = 1, \dots, n$).

$\Delta_h^{2\omega_n}$ denotes the multiple difference of the second order in x_j with steps h_j for $j \in e$. The constant $M_p^{(\mathbf{r}^e)}(f)$ does not depend on \mathbf{h} .

We put

$$\|f\|_{S_p^{(r)}} = \|f\|_{S_p^{(r)}H(R_n)} = \sum_{e \subset e_r} M_p^{(\mathbf{r}^e)}(f), \quad M_p^{(\mathbf{r}^0)}(f) = \|f\|_{L_p(R_n)}.$$

Theorem 1. A function $f \in S_p^{(r)} H(R_n) = S_p^{(r)}$ ($1 \leq p \leq \infty$) can be represented in the form

$$f(\bar{x}) = \sum_{e \subset e_r} \sum_{k^e \geq 0} Q_{k^e}(\bar{x}), \quad (2)$$

* This class is, in a certain sense, close to the class $S_p^{(r)}$, which was considered in my note (8).

where the first sum extends over all subsets $e \subset e_r$, including $e = 0$, $e = e_r$. The second sum extends over all possible integral nonnegative vectors $k^e = (k_1^e, \dots, k_n^e)$ ($k_j^e = 0$ for $j \in e_n - e$). The functions Q_{k^e} are entire functions of exponential type of degrees $2^{k_j^e}$ in x_j , $j \in e_r$ (thus, of degree 1 in x_j , $j \in e_r - e$), satisfying the inequalities

$$\|Q_{k^e}\|_{L_p(R_n)} \leq \frac{c \|f\|_{S_p^{(r)} H(R_n)}}{2^{\sum_{j \in e} k_j^e r_j}}, \quad (e \subset e_r). \quad (3)$$

Theorem 2 (converse to Theorem 1). Let a function $f(\bar{x})$, defined on R_n , be representable in the form (2) with terms Q_{k^e} possessing the properties listed in Theorem 1, where in (3) it is only necessary to replace $\|f\|_{S_p^{(r)} H(R_n)}$ by M .

Then $f \in S_p^{(r)} H(R_n)$ and the inequality

$$\|f\|_{S_p^{(r)}(R_n)} \leq cM,$$

holds, where c does not depend on M .

The proof of Theorem 1 is based on the introduction of an entire function $g(t)$ of degree 1, for which

$$\int_{-\infty}^{\infty} g(t) dt = 1, \quad \int_{-\infty}^{\infty} |t|^s |g(t)| dt \leq b^s \quad (s = 0, 1, \dots, \rho),$$

where $\rho > 0$ is a given natural number. Let m be a natural number, $1 \leq m \leq n$, and let $\bar{\omega}_m = (\underbrace{1, \dots, 1}_{m \text{ times}})$. Put, for $\bar{\mu} = (\mu_1, \dots, \mu_m)$ ($\mu_j > 0$),

$$\sigma_{\bar{\mu}}(\bar{x}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{j=1}^m g(u_j) \Delta_{\frac{\mu_1}{\mu_1}, \dots, \frac{\mu_m}{\mu_m}}^{(\rho+2)\bar{\omega}_m} f(\bar{x}) du_1 \dots du_m, \quad (4)$$

where the multiple difference has order $\rho + 2$ in each variable x_1, \dots, x_m , respectively, with steps

$$\frac{u_1}{\mu_1}, \dots, \frac{u_m}{\mu_m}.$$

Next, if we set

$$\tau_{\mathbf{k}}(\bar{x}) = \Delta_{1, \dots, 1}^{\bar{\omega}_m} \sigma_{2^{k_1}, \dots, 2^{k_m}} \quad (k_j = 0, 1, \dots; j = 1, \dots, m),$$

where $\Delta_{1, \dots, 1}^m$ denotes the m -fold difference of first order in each of the variables k_j with step 1, then it turns out that

$$f(\bar{x}) = \sum' \gamma_{\mathbf{i}}(x) + \sum_{\mathbf{k} \geq 0} \tau_{\mathbf{k}}(\bar{x}), \quad 0 \leq \mathbf{i} \leq (\rho + 2)\bar{\omega}_m, \quad \mathbf{k} = (k_1, \dots, k_m), \quad (5)$$

where \sum' extends over all integral nonnegative, but nonzero, vectors \mathbf{i} for which $0 \leq \mathbf{i} \leq (\rho + 2)\bar{\omega}_m$, and the sum $\sum_{\mathbf{k}}$ is an infinite m -fold series.

If a function $f \in L_p(R_n)$ is such that

$$\|\sigma_{\bar{\mu}}\|_p \rightarrow 0 \quad \text{as} \quad \sum_1^m \mu_j^2 \rightarrow 0, \quad (6)$$

then the series $\sum_{\mathbf{k}}$ converges in the sense of $L_p(R_n)$. For finite p , condition (6) follows automatically from the fact that $f \in L_p(R_n)$.

The following properties hold:

- 1) $\|\gamma_{\mathbf{i}}\|_p, \|\tau_{\mathbf{k}}\|_p \leq c\|f\|_p$.
- 2) $\gamma_{\mathbf{i}}(\bar{x})$ is entire of degree 1 in $x_j, j \in e_i$, and in those $x_j, j \in e_n - e^m$, with respect to which f is entire of degree 1.
- 3) $\tau_{\mathbf{k}}$ are entire of degrees 2^{k_j} in $x_j, j \in e_m$, and of degree 1 in those $x_j, j \in e_n - e_m$, with respect to which f is entire of degree 1.
- 4) If $f \in S_p^{(r)}, e_r \subset e_m, 0 \leq r \leq \rho\bar{\omega}_m$, then $\gamma_{\mathbf{i}} \in S_p^{r(e_r - e_i)}$ and

$$\|\gamma_{\mathbf{i}}\|_{S_p^{r(e_r - e_i)}} \leq c\|f\|_{S_p^{(r)}}.$$

- 5) If $f \in S_p^{(r)}, e_r \subset e_m, 0 < r \leq \rho\bar{\omega}_m$, then

$$\|\tau_{\mathbf{k}}\|_p \leq \frac{c\|f\|_{S_p^{(r)}}}{2^{\sum_{j \in e_r} k_j r_j}}. \quad (7)$$

To obtain Theorem 1, one must apply equality (5) with $e_r = e_m$, and then from $f(\bar{x})$ there is separated off the sum $\sum_{\mathbf{k}}$, which in (2) corresponds to $e \in e_r$. To the functions $\gamma_{\mathbf{i}}$ Theorem 1 is again applied, where the role of e_m must now be played by the set $e_r - e_i$. As a result, for each i a series $\sum_{\mathbf{k}^e} \tau_{\mathbf{k}^e}$ with $e = e_r - e_i$ is separated off. After a finite number of operations the process of separation terminates and leads to the series (2).

Theorem 2 is proved in the spirit of the converse theorem of approximation of S. N. Bernstein. We give several inequalities that can be obtained as consequences of Theorems 1 and 2.

If $1 \leq p \leq p' \leq \infty$, $\vec{\rho} = (\rho_1, \dots, \rho_n) = r - \left(\frac{1}{p} - \frac{1}{p'}\right) \vec{\omega}_n$, $\rho_j > 0$ ($j = 1, \dots, n$), then*

$$S_p^{(r)}(R_n) \rightarrow S_p^{(\vec{\rho})}(R_n). \quad (8)$$

If

$$e^1 + e^2 = e_r, \quad e^1 e^2 = 0, \quad e_n = e_r + e^3, \quad r_j - \frac{1}{p} > 0, \quad j \in e^2,$$

then

$$S_p^{(r)}(R_n) \rightarrow S_p^{(re^1 + e^3)}(R_{e^1 + e^3}), \quad (9)$$

where $R_{e^1 + e^3}$ is the subspace of points $\vec{x} = (x_1, \dots, x_n)$ with fixed $x_j = x_j^0$, $j \in e^2$.

Explanation. Represent the function $f \in S_p^{(r)}(R_n)$ in the form (2), and estimate the norms $\|Q_{k^e}\|_{p'}$, by already known inequalities (see (1) (3) or (2) (1,10)), through the norms $\|Q_{k^e}\|_p$, whose estimates are expressed by inequalities (3). Then we apply the converse theorem and obtain (8). We proceed in the same way in the case (9), but now the $\|Q_{k^e}\|_{L_p(R_{e^1 + e^3})}$ are estimated by other known inequalities through $\|Q_{k^e}\|_{L_p(R_n)}$ (see (2), (1.12)). It is also necessary to replace all the series (2) by series of smaller multiplicity, summing them with respect to the indices k_j^e , $j \in e^2$.

The following assertion holds:

If $f \in S_p^{(r)}(R_n) S_p^{(\vec{\rho})}(R_n)$, $\lambda + \mu = 1$, $\lambda, \mu \geq 0$, then $f \in S_p^{(\lambda r + \mu \vec{\rho})}(R_n)$ and

$$\|f\|_{S_p^{(\lambda r + \mu \vec{\rho})}} \leq c \|f\|_{S_p^{(r)}}^\lambda \|f\|_{S_p^{(\vec{\rho})}}^\mu. \quad (10)$$

* For normed spaces E, E_1 we write $E \rightarrow E_1$ if $E \subset E_1$, and $\|f\|_{E_1} \leq c \|f\|_E$ for all $f \in E$.

Explanation. The theorem (10) is first proved when $e_r = e_{\vec{\rho}} = e'$, where e' consists of a single index. Suppose now that (10) holds when $e_r, e_{\vec{\rho}} \subset e_{m-1}$, and it is given that $e_r + e_{\vec{\rho}} \subset e_m = e_{\lambda r + \mu \vec{\rho}}$. We apply to f the expansion (5), in which, therefore, simultaneously

$$\|\tau_k\|_\rho \leq \frac{c_1 \|f\|_{S_p^{(r)}}}{2^{\sum_{j \in e_r} k_j r_j}}, \quad \|\tau_k\|_\rho \leq \frac{c_1 \|f\|_{S_p^{(\vec{\rho})}}}{2^{\sum_{j \in e_{\vec{\rho}}} k_j \rho_j}}. \quad (11)$$

Raising the first and second inequalities respectively to the powers λ, μ , and then multiplying, we obtain

$$\|\tau_k\|_\rho \leq \frac{c_1 \|f\|_{S_\rho^{(r)}}^\lambda \|f\|_{S_\rho^{(\bar{\rho})}}^\mu}{2^{\sum_{j \in e_m} k_j (\lambda r_j + \mu \rho_j)}}.$$

As a result, the sum of the series with terms τ_k , by Theorem 2, belongs to $S_\rho^{(\lambda r + \mu \bar{\rho})}$. Further (by property 4)),

$$\gamma_i \rightarrow \in S_\rho^{(r e_r - e_i)} S_\rho^{(\bar{\rho} e_{\bar{\rho}} - e_i)},$$

$(e_r - e) + (e_{\bar{\rho}} - e_i) \subset e_m - e_i$, and since $e_i \neq 0$, because $i \neq 0$, it follows, in view of the fact that the assertion is true for $m - 1$, that

$$f \in S_\rho^{(q)}, \tag{12}$$

where

$$q = \lambda r^{(e_r - e_i)} + \mu \rho^{(e_{\bar{\rho}} - e_i)} = (\lambda r + \mu \rho)^{(e_m - e_i)}.$$

Moreover, f is an entire function of degree 1 with respect to x_j , $j \in e_i$, and this, together with (12), implies that $f \in S_\rho^{\lambda r + \mu \bar{\rho}}$.

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