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Abstract

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CHARACTERIZATION OF BOUNDARY VALUES OF FUNCTIONS FROM $L_p^r(E_n)$ ON HYPERPLANES

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1. The scales of differentiable functions of many variables $B_p^r(E_n)$ and $L_p^r(E_n)$ are constructed on the basis of taking into account the differential properties of functions in the metrics L_p. It is known ([1,2]) that the spaces $B_p^r(E_n)$ possess a well-ordered system of embedding theorems*, remarkable, in particular, for its closedness with respect to boundary embeddings. In the author' s preceding paper ([4]), analogous theorems were obtained for the spaces $L_p^r(E_n)$, with the sole exception of the boundary embedding theorems just mentioned. In the present paper the boundary values of functions from $L_p^r(E_n)$ are characterized in terms of $B_p^{r'}$-spaces. This complication of the relations (a departure from the scale!) is compensated by the greater precision of the assertions not only for the L_p^r-spaces themselves, but also for the spaces B_p^r. It turns out, for example, that for $p > 2$ one can ensure a better quality of the extended function than is guaranteed in the theory of B_p^r-spaces.

2. According to the initial definition ([4]), the space $L_p^r(E_n)$, $1 < p < \infty$, $r > 0$, consists of functions $f(x)$, $x = (x_1, \dots, x_n) \in E_n$, summable with power p in E_n together with their unmixed generalized Liouville derivatives of order r. It was shown that the membership of $f(x)$ in $L_p^r(E_n)$ is equivalent to its representability by a potential of the form

$$f(x) = \int_{E_n} G_r^n(x-y)\,\varphi(y)\,dy, \qquad \varphi(x) \in L_p(E_n), \tag{1}$$

where

$$G_r^n(x) = \frac{|x|^{(r-n)/2} K_{(n-r)/2}(|x|)}{2^{(n+r-2)/2}\pi^{n/2}\Gamma(r/2)};$$

$K_\nu(t)$ is the Macdonald function. The norm of f in $L_p^r(E_n)$ is taken to be $\|\varphi\|_{L_p(E_n)}$.

The space $B_p^r(E_n)$, $1 \le p < \infty$, $r > 0$ ([1,2]), is the closure of the set of smooth finite functions with respect to the norm

$$\|f\|_{B^r_p(E_n)} = \|f\|_{L_p(E_n)} + \sum_{i=1}^{n} \left\{ \int_{E_n} \int_{E_n} \frac{\left| f_i^{(\bar r)}(x) - 2 f_i^{(\bar r)}(\frac{x+y}{2}) + f_i^{(\bar r)}(y) \right|^p}{|x-y|^{n+p(r-\bar r)}} \, dx \, dy \right\}^{1/p},$$

where $\bar r$ is the greatest integer less than r, and $f_i^{(\bar r)} = \partial^{\bar r} f / \partial x_i^{\bar r}$, $i = 1, \dots, n$.

The relation between the spaces $B^r_p(E_n)$ and $L^r_p(E_n)$ is given by the theorem:

Theorem 1**. The following continuous embeddings hold

a) $B^r_p(E_n) \subset L^r_p(E_n), \quad 1 < p \le 2;$

b) $L^r_p(E_n) \subset B^r_p(E_n), \quad 2 \le p < \infty.$

3. Another aspect of the connection between the spaces $L^r_p(E_n)$ and $B^r_p(E_n)$ is the somewhat paradoxical fact that the boundary values of functions from $L^r_p(E_n)$ (as well as of functions from $B^r_p(E_n)$!) are conversely characterized in terms of $B^{r'}_p(E_m)$-spaces. By the **trace** of $f(x)$ on the hyperplane E_m we shall mean the collection of limiting values $\tilde f(x)$

* Similar theorems in such a complete form were first given by S. M. Nikol' skii ([3]) for H^r_p-classes.

** This result was reported by the author in Baku at the Second All-Union Conference on the Constructive Theory of Functions in October 1962.

as x tends to E_m. Under our conditions the trace of a function is at the same time also the boundary value in the sense of convergence in the p-mean.

Theorem 2. I. If a function $f(x) \in L^r_p(E_n)$, then its trace on the hyperplane of m dimensions forms a function f' belonging to $B^{r'}_p(E_m)$,

$$r' = r - \frac{n-m}{p},$$

and, moreover, the norm of f' in $B^{r'}_p(E_m)$ is majorized by the norm

$$\|f\|_{L^r_p(E_n)}.$$

II. If on the hyperplane E_m a function $f' \in B^{r'}_p(E_m)$ is given, then there exists a function $f(x) \in L^r_p(E_n)$ having f' as its trace on E_m, and the norm of f in $L^r_p(E_n)$ is majorized by the norm $\|f'\|_{B^{r'}_p(E_m)}$.

The proof of both parts of the theorem may be carried out for the case $m = n-1$; the further reduction is effected on the basis of embedding theorems for B^r_p-spaces ([1,2]). Consider case I. Extend the function $f \in L^r_p(E_n)$ metaharmonically* into the half-space $E^+_{n+1}\{(x,t);\, x \in E_n,\ t > 0\}$ in the form

$$u(x,t) = \frac{2t}{(2\pi)^{(n+1)/2}} \int_{E_n} \frac{K_{(n+1)/2}(\sqrt{|x-y|^2 + t^2})}{(|x-y|^2 + t)^{(n+1)/4}} \, f(y) \, dy.$$

Using representation (1), we obtain

$$u(x,t) = \int_{E_n} N_n(x-y,t)\varphi(y)\,dy,$$

$$N_n(x,t) = \frac{2}{(2\pi)^{(n+1)/2}\Gamma(r)} \int_0^\infty \frac{(t+s)s^{r-1}K_{(n+1)/2}\left(\sqrt{|x|^2+(t+s)^2}\right)}{\{|x|^2+(t+s)^2\}^{(n+1)/4}}\,ds.$$

For the kernel $N_n(x,t)$ the estimates

$$|D^k N_n(x,t)| \le \frac{c}{(|x|^2+t^2)^{(n-r+l)/2}},$$

are valid, where $D^k N_n(x,t)$ is any derivative of N_n of order k, $0 \le k \le l$. On the basis of these estimates, with the aid of the generalized Minkowski inequality and Theorem 319 from ([5]), one obtains the inequality

$$\int_0^\infty t^\beta\,dt \int_{E_{n-1}} \left\{ \sum_{l_1+\cdots+l_{n-1}+m=l} \left| \frac{\partial^l u(x_1,\dots,x_{n-1},0,t)}{\partial x_1^{l_1}\cdots\partial x_{n-1}^{l_{n-1}}\partial t^m} \right|^p + |u|^p \right\} dx' \le c\|\varphi\|_{L_p}^p, \tag{*}$$

where E_{n-1} is the hyperplane of the variables $(x_1,\dots,x_{n-1}) \equiv x'$, $x_n = 0$,

$$\beta = (p-1)-(\gamma-[\gamma])p, \qquad l = [\gamma]+1, \qquad [\gamma] \text{ is the integer part}, \qquad \gamma = r-\frac{1}{p}.$$

But the finiteness of the weighted integral $(*)$ ensures that the function

$$\lim_{t\to 0} u(x_1,\dots,x_{n-1},0,t) = f'(x')$$

belongs to $B_p^{r-1/p}(E_{n-1})$ and gives the corresponding estimate. It is also clear that

$$f'(x') = \lim_{x_n\to 0} f(x_1,\dots,x_n),$$

since this last limiting passage can be replaced by a limiting passage in the function $u(x,t)$ along a suitable Π-shaped contour for $t > 0$.

Let us pass to the second part of Theorem 2. The definition of $L_p^r(E_n)$ may be expressed by the requirements

$$f \in L_p(E_n);$$

$$\frac{\partial^{[r]+1}}{\partial x_i^{[r]+1}}(-\Delta_{x'}+1)^{-\alpha/2} f \in L_p(E_n), \qquad i = 1,\dots,n-1;$$

$$\frac{\partial^{[r]+1}}{\partial x_n^{[r]+1}}\left(-\frac{\partial^2}{\partial x_n^2}+1\right)^{-\alpha/2} f \in L_p(E_n), \qquad \alpha = [r]+1-r,$$

* That is, by the solution of the problem $\Delta u - u = 0, \quad u(x,0) = f(x)$.

where

$$(-\Delta_{x'} + 1)^{-\alpha/2} f = \int_{E_{n-1}} G_\alpha^{n-1}(x' - y') f(y', x_n)\, dy',$$

$$\left(-\frac{\partial^2}{\partial x_n^2} + 1\right)^{-\alpha/2} f = \int_{-\infty}^{\infty} G_\alpha^1(x_n - y_n) f(x', y_n)\, dy_n.$$

Now let $f'(x') \in B_p^{r'}(E_{n-1})$, $r' = r - 1/p$. Take the metaharmonic extension $u(x', x_n)$ of the function $f'(x')$ into the half-space E_n^+ $(x_n > 0)$ and extend it by the method of Whitney and Hestenes, preserving smoothness, to the whole space E_n. The function $U(x)$ thus obtained gives the desired extension. The proof is reduced to estimating the integral

$$\int_{E_n} \left\{ |U|^p + \left| \frac{\partial^{[r]+1}}{\partial x_n^{[r]+1}} \left(-\frac{\partial^2}{\partial x_n^2} + 1\right)^{-\alpha/2} U \right|^p + \sum_{i=1}^{n} \left| \frac{\partial^{[r]+1}}{\partial x_i^{[r]+1}} (-\Delta_{x'} + 1)^{-\alpha/2} U \right|^p \right\} dx,$$

which is obtained on the basis of Hilbert' s two-parameter inequality and Theorem 329 from ([5]). Various properties of metaharmonic extension are also used, in particular the equality

$$\int_{E_{n-1}} G_\alpha^{n-1}(x' - y') u(y', x_n)\, dy' = \frac{1}{\Gamma(\alpha)} \int_{x_n}^{\infty} \frac{u(x', \tau)\, d\tau}{(\tau - x_n)^{1-\alpha}}, \qquad x_n > 0.$$

In the intermediate computations there occur weighted integrals of $u(x', x_n)$.

Finally, let us formulate a general theorem on the extension of a system of functions. Let E_m be the hyperplane obtained by fixing the last $n - m$ coordinates; for definiteness we shall assume $x_{m+1} = 0, \dots, x_n = 0$.

Theorem 3. Let on the hyperplane E_m there be given a system of functions

$$f^\lambda_{\lambda_{m+1},\dots,\lambda_n}(x_1, \dots, x_m) \in B_p^{r-\lambda-(n-m)/p}(E_m),$$

$$\lambda = 0, \dots, l; \qquad r - \lambda - \frac{n-m}{p} > 0; \qquad \lambda_i \geq 0, \quad \lambda_{m+1} + \dots + \lambda_n = \lambda.$$

Then there exists a function of n variables $f(x)$ having the properties:

$$\text{a)}\quad f(x) \in L_p^r(E_n); \qquad \text{b)}\quad \left.\frac{\partial^\lambda f}{\partial x_{m+1}^{\lambda_{m+1}} \cdots \partial x_n^{\lambda_n}}\right|_{E_m} = f^\lambda_{\lambda_{m+1},\dots,\lambda_n}(x_1, \dots, x_m);$$

$$\text{c)}\quad \|f\|_{L_p^r(E_n)} \leq c \sum_{\lambda=0}^{l} \sum_{\lambda_{m+1}+\dots+\lambda_n=\lambda} \left\| f^\lambda_{\lambda_{m+1},\dots,\lambda_n} \right\|_{B_p^{r-\lambda-\frac{n-m}{p}}(E_m)}.$$

The proof is carried out according to the schemes of work ([6]).

Let us note in conclusion that, on the whole, the methods of the present paper rely essentially on the use of weighted spaces, following the example of L. D. Kudryavtsev ([8]). Let us also recall (see ([4])) that the spaces $L_p^r(E_n)$ give a natural extension of the $W_p^{(l)}$-classification of function spaces according to S. L. Sobolev ([9]) to noninteger indices of differentiation; in particular (for integer $r = l$),

$$L_p^l(E_n) \equiv W_p^{(l)}(E_n).$$

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Note: Figure translations are in progress. See original paper for figures.