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Abstract

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Quotient Spaces of the Idèle Group of an Algebraic Group and Sheaf Cohomology

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Let $G \subset GL(n,k)$ be an algebraic group over a field k of algebraic numbers; $J(G)$ its idèle group; $J_0(G)$ the subgroup of unit idèles ([1]). An important arithmetical characteristic of the group G is the double quotient space

$$G \backslash J(G)/J_0(G),$$

studied in ([1–3]). In the present paper another approach is proposed to the study of these quotient spaces, realizing them as cohomology sets of certain sheaves. Along this path it has proved possible to obtain, in a comparatively simple way, both some known facts and several new ones.

1. Let k be a finite algebraic extension of the field of rational numbers Q; p a prime divisor of the field k; J the ring of integers of the field k; J_p the ring of p-integral elements of the field k. $GL(n,k)$, $GL(n,J)$, and $GL(n,J_p)$ are the full linear groups over the field k, the rings J and J_p, respectively. Clearly,

   $$GL(n,J) \subset GL(n,J_p) \subset GL(n,k).$$

   Let

   $$GL(n,\nu,J_p) = \{s \in GL(n,J_p),\ s \equiv I \pmod{p^\nu},\ \nu \geqslant 0\}$$

   be the congruence subgroup of $GL(n,J_p)$ modulo p^ν. Let now $G \subset GL(n,k)$ be an algebraic group over the field k ([4]). Put

   $$E_p^\nu = G \cap GL(n,\nu,J_p)$$

   and call the group $E_p = E_p^0$ the group of local units of the group G at the point p. E_p^ν is a normal divisor of E_p of finite index. The following two lemmas are obvious.

Lemma 1. Let $s \in G$; then

$$E_p^\nu \subset s^{-1}E_p^\lambda s$$

for all $\nu \geqslant \nu_0 \geqslant \lambda$.

Lemma 2. The groups E_p^ν and $s^{-1}E_p^\nu s$ are commensurable.

Let k_p be the p-adic completion of the field k, and let $G_p \subset GL(n,k_p)$ be the group obtained from G by extending the ground field k to k_p ([4]). The groups $\bar{E}_p^\nu$ and $G \subset G_p$, $E_p^\nu \subset \bar{E}_p^\nu$, are defined in a completely analogous way. The group

G_p is locally compact in the p-adic topology, and the groups $\bar{E}_p^\nu$ are compact. Let

$$\varphi : G/E_p \to G_p/\bar{E}_p$$

be the natural mapping of quotient spaces. It is easy to see that φ is injective. We shall show that φ is bijective. This follows from a more general result for connected groups G:

Theorem 1 (independence theorem). Let

$$\Pi = \prod_{i=1}^{k} G_{p_i}$$

be the direct product of topological groups G_{p_i}, and let

$$\varphi : G \to \Pi$$

be the natural embedding map. Then $\varphi(G)$ is everywhere dense in Π.

The theorem is equivalent to the solvability of the system of congruences

$$s \equiv s_i \pmod{p_i^{\nu_i}} \qquad (1 \leqslant i \leqslant k)$$

in the group G for arbitrary $s_i \in G_{p_i}$ and $\nu_i \geqslant 0$. The group G has a rational parametric representation over the field k ([5]), and the s_i can always be chosen so that they are covered by the given parametric representation. From this the assertion of the theorem follows easily.

2. Let X be the set of all simple finite divisors of the field k. We define a topology on the set X, taking as closed sets the finite subsets $F \subset X$ and X itself (the Zariski topology). Let

   $$E_U = \bigcap_{p \in U} E_p,$$

   where U is an open subset of X. If $V \subset U$, then

   $$f_U^V : E_U \to E_V$$

   is the natural embedding map. The family $\{E_U, f_U^V\}$

defines a sheaf of groups over X ([6]). We shall call it the sheaf of local units of the group G over X and denote it by $E(G)$. Let $m = p_1^{\nu_1} p_2^{\nu_2} \cdots p_k^{\nu_k}$ be an integral divisor of the field k. Replacing in the sheaf $E(G)$ a finite number of layers E_{p_i} by $E_{p_i}^{\nu_i}$ $(1 \leq i \leq k)$, we obtain a subsheaf $E(G, m)$ of the sheaf $E(G)$. We shall call it the ray sheaf of the group G mod m. By the method described in ([7]), the cohomology sets $H^0(X, E)$ and $H^1(X, E)$ are constructed. Obviously, $H^0(X, E(G)) = G \cap GL(n, J)$. For the interpretation of $H^1(X, E(G))$, consider the exact sequence $1 \to E(G) \to G \to G/E(G) \to e$, where G is the constant sheaf over X, and the exact cohomology sequence generated by it: $1 \to$

$H^0(X,E(G)) \to H^0(X,G) \to H^0(X,G/E(G)) \to H^1(X,E(G)) \to H^1(X,G)$. $H^0(X,G) = G$, $H^1(X,G) = e$, since G is a constant sheaf. Then $H^1(X,E(G))$ is bijective to the set of classes of intransitivity of the group G, acting on the set $H^0(X,G/E(G))$. The set $H^0(X,G/E(G))$ is the collection of "divisors" of the algebraic group G, and it is not difficult to verify, taking Theorem 1 into account, that $H^0(X,G/E(G))$ is bijective to the quotient space $J(G)/J_0(G)$. Thus we obtain the following assertion:

Theorem 2. *The set $H^1(X,E(G))$ is bijective to the double quotient space $G\backslash J(G)/J_0(G)$.*

Remark. The quotient space $G\backslash J(G)/J_0(G)$ was considered by T. Ono ([1,2]), who conjectured that this space is finite. Recently this conjecture was proved by A. Borel ([3]). Theorem 2 shows then, on the one hand, that $H^1(X,E(G))$ is finite for any algebraic group, and, on the other hand, allows one to use the technique of cohomology to study the indicated quotient spaces.

Let A be a subsheaf of the constant sheaf G, containing some ray sheaf $E(G,m)$. We shall call sheaves isomorphic to sheaves A admissible. Theorem 1 shows that the sheaf $G/E(G,m)$ is flasque ([6]). Hence follows

Proposition 1. *Let A be an admissible sheaf. Then the natural mapping*

$$H^1(X,E(G,m)) \to H^1(X,A)$$

is surjective.

Theorem 3. *Let $G \subset GL(n,k)$ be a unipotent algebraic group and A an admissible subsheaf of G. Then $H^1(X,A) = e$.*

According to Proposition 1, it suffices to consider the case $A = E(G,m)$. Let $\dim G = 1$. Then G is commutative and there exists a polynomial isomorphism φ between G and the additive group of the field k ([8]). The mapping φ induces an isomorphism between $E(G,m)$ and the sheaf of ideals F of the field k, but F is coherent and $H^1(X,F) = 0$. Let $\dim G = r > 1$. Then G is representable as a semidirect product of its subgroups of smaller dimension. The method of induction proves the theorem.

3. Let B be a sheaf of groups over an arbitrary space X, and A its subsheaf. $\mathfrak{U} = \{U_i\}_{i\in I}$ is an open covering of X, and $z = \{z_{ij}\}$ is a cocycle of the covering $\mathfrak{U}$ with values in B, with $z_{ij}^{-1}A_{ij}z_{ij} = A_{ij}$ (A_k is the restriction of the sheaf A to U_k). Further, let A^z be the sheaf glued from the sheaves A_k by means of the cocycle z ([7]). In ([7]) the following result is contained implicitly:

Proposition 2. *Let*

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

be an exact sequence of sheaves of groups, and suppose every section of the sheaf C over an arbitrary open set $U \subset X$ is the image of some section of the sheaf

B over U. In order that the mapping $i_1 : H^1(X, A) \to H^1(X, B)$ be injective, it is sufficient that $H^1(X, A^z) = e$ for all cocycles z of the sheaf B.

4. **Proposition 3.** *Let $G \subset GL(n, k)$ be an algebraic group, $\mathfrak{U} = \{U_i\}$ an open finite covering of X, $z = \{z_{ij}\}$ a cocycle of the covering $\mathfrak{U}$ with values in the sheaf $E(GL)$, with $z_{ij}^{-1} G z_{ij} = G$. Let A be an admissible subsheaf of G; then A^z is also an admissible subsheaf of G.*

Let V be an arbitrary open subset of X. Every section s of the sheaf A^z over V can be identified with a system $\{s_k\}$, where $s_k \in \Gamma(V \cap U_k, A_k)$ and $s_k = z_{ik}^{-1} s_i z_{ik}$. Since $z_{ik} = z_{i1} z_{k1}^{-1}$, it follows that $z_{k1}^{-1} s_k z_{k1} = z_{i1}^{-1} s_i z_{i1} = s \in G$. The mapping $\{s_k\} \mapsto s$ defines a monomorphism of the sheaf A^z into the constant sheaf G. The fact that A^z contains a ray sheaf $E(G, m)$ for some m follows from Lemma 1.

Theorem 4. Let $G \subset GL(n, k)$ be a connected algebraic group, and let $G = NA$ be its decomposition into a semidirect product, where N is unipotent and A is reductive (Chevalley decomposition ([5])). Then there exists a bijection $H^1(X, E(G)) \leftrightarrow H^1(X, E'(A))$, where $E'(A)$ is a subsheaf of the constant sheaf A containing $E(A)$.

In the exact sequence $1 \to N \to G \xrightarrow{\varphi} A \to 1$, the mapping $\varphi : s = na \mapsto a$ is polynomial and $\varphi(E_p^G) \supset E_p^A$. We have an exact sequence of sheaves $1 \to E'(N) \to E(G) \to E'(A) \to 1$, where $E'(N) \supset E(N)$, $E'(A) \supset E(A)$. Hence $H^1(X, E'(N)) \to H^1(X, E(G)) \xrightarrow{\varphi_1} H^1(X, E'(A)) \to e$. The mapping φ_1 is a bijection according to Theorem 3 and Propositions 2 and 3.

Corollary. Let $G \subset GL(n, k)$ be a connected solvable algebraic group. Then on the set $H^1(X, E(G))$ one can define the structure of a commutative group.

Indeed, in the decomposition $G = NA$, A is a commutative group ([8]).

5. Example 1. G is a connected algebraic diagonal group, i.e. $G \subset D(n) \subset GL(n, k)$, and $\dim G = r$. Then $H^1(X, E(G)) \simeq h^r$, where h is the ideal class group of the field k.

Example 2. The group $T(n) \subset GL(n, k)$ of all triangular matrices. $H^1(X, E(T(n))) \simeq h^n$.

Example 3. The groups $GL(n, Q)$ and $SL(n, Q)$. There are decompositions $GL(n, Q) = T(n) GL(n, J_p)$, $SL(n, Q) = T^0(n) SL(n, J_p)$, $T^0(n) = T(n) \cap SL$. Hence $H^1(X, E(GL)) = H^1(X, E(SL)) = e$. Decompositions of this type are known for semisimple Chevalley groups ([9]). The sets $H^1(X, E(G))$ for the indicated groups are trivial.

6. Let $G \subset GL(n, k)$ be a reductive algebraic group. Extend the field k to the field C of complex numbers. The homogeneous space $GL(n, C)/G^C$ is an affine variety defined over k, and there exists a rational representation $\pi : GL(n, C) \to GL(V)$, defined over k, and a point $v \in V_k$ such that the orbit $v \circ \pi(GL(n, C))$ is closed and the isotropy group $G_v = G^C$ ([10,11]). Then the homogeneous space

$GL(n,k)/G$ can be identified with the subset V_k, namely, $GL(n,k)/G \leftrightarrow v \circ \pi(GL(n,k)) = M$, and $GL(n,J_p)/E_p \leftrightarrow v \circ \pi(GL(n,J_p)) = M_p \subset M$. Let $k = Q$. One can choose $v \in V_Z$ so that $\pi(GL(n,Z)) \subset GL(V,Z)$. Then we have exact sequences: $1 \to G \to GL(n,k) \to M \to e$, $1 \to E(G) \to E(GL) \to E(M) \to e$, and

$$1 \to H^0(E(G)) \to H^0(E(GL)) \to H^0(E(M)) \to H^1(E(G)) \to H^1(E(GL)). \tag{1}$$

$H^1(X,E(GL)) = e$ according to § 5, Example 3; $H^0(X,E(GL)) = GL(n,Z)$; $H^0(X,E(M)) = \bigcap_{p\in X} M_p \subseteq M_Z$. In works ([10,11]) it is proved that M_Z consists of a finite number of orbits of the group $GL(n,Z)$. It then follows from (1) that

Theorem 5. The set $H^1(X,E(G))$ for an algebraic reductive group $G \subset GL(n,Q)$ is bijective to the set of orbits of the group $GL(n,Z)$ acting on $M_Z = \bigcap_{p\in X} M_p \subseteq M_Z$ and, consequently, is finite.

Let $\{\omega_1, \omega_2, \dots, \omega_m\}$ be an integral basis of the field k, $m = [k:Q]$, and let $\mu : k \to M(m,Q)$ be the regular representation of the field k over Q relative to

of this basis. The mapping μ defines a monomorphism $\mu^* : G \to GL(nm,Q)$ and $H^i(X,E(G)) = H^i(Y,E(G^*))$, where X and Y are the spaces of prime divisors of the fields k and Q, respectively, $G \subset GL(n,k)$, $G^* = \mu^*(G)$, $i = 0,1$. If, for example, one takes the quadratic field $k = Q(\sqrt{d})$, then $\mu(k^*) = G \subset GL(2,Q)$, and G is a Q-torus. Theorems 2 and 5 give the well-known relation between the ideal class group of the field k and the classes of equivalent quadratic forms of determinant d. From the indicated reduction, taking Theorems 2, 4, and 5 into account, it follows that

Theorem 6 (Borel ([3])). *The set $G\backslash J(G)/J_0(G)$ is finite for any algebraic group $G \subset GL(n,k)$.*

Remark. This theorem can probably be proved by using the finiteness of the number of ideal classes of algebraic fields and decompositions of type p. 5, Example 3.

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Note: Figure translations are in progress. See original paper for figures.