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Abstract

Full Text

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INEQUALITIES FOR POLYNOMIALS AND RATIONAL FUNCTIONS

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In 1912 S. N. Bernstein ([1]) proved that if the polynomial $P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ of degree n satisfies the inequality $|P_n(x)| \leqslant L$ on the interval $[-1, 1]$, then at every point x of the real axis outside the interval $[-1, 1]$ one has

$$|P_n(x)| \leqslant L \left| \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2} \right|. \tag{1}$$

A further generalization of this assertion, also due to S. N. Bernstein ([2]), played an essential role in the study of best approximation of analytic functions by means of polynomials.

In this note, by simple considerations, inequality (1) is refined in a corresponding way for polynomials having no zeros in a given disk, and other inequalities for polynomials and rational functions are proved.

Let $z_j = r_{je}^{i\varphi_j}$ $(j = 1, 2, \dots, n)$ be the zeros of the polynomial $P_n(x)$. Put

$$M(P_n, a) = \max_{-a \leqslant x \leqslant a} |P_n(x)|.$$

Theorem 1. If $a \geqslant 1$ and $r_i \geqslant \sqrt{a} + \gamma$ $(\gamma \geqslant 0;\ j = 1, 2, \dots, n)$, then the inequality

$$M(P_n, a) \leqslant a^{n/2} \left(\frac{1+\sqrt{a}+\gamma\sqrt{a}}{1+\sqrt{a}+\gamma} \right)^n M(P_n, 1) \tag{2}$$

holds for the polynomial $P_n(x)$.

Proof. Let $M(P_n, a) = |P_n(x_0)|$, where $-a \leqslant x_0 \leqslant a$. Then we have

$$\frac{M(P_n, a)}{M(P_n, 1)} \leqslant \left| \frac{P_n(x_0)}{P_n(x_0/a)} \right| = \prod_{j=1}^{n} \left| \frac{x_0 - r_{je}^{i\varphi_j}}{x_0/a - r_{je}^{i\varphi_j}} \right|,$$

where $-1 \leqslant x_0/a \leqslant 1$. Write this formula in the following form:

$$\frac{M(P_n, a)}{M(P_n, 1)} \leqslant a^n \prod_{j=1}^{n} \sqrt{U_j(\varphi_j)}.$$

If $-a \leqslant x_0 \leqslant 0$, then the relation

$$U_j(\varphi_j) - \left(\frac{r_j - x_0}{ar_j - x_0}\right)^2 = \frac{2x_0 r_j (a-1)(ar_j^2 - x_0)(1 - \cos\varphi_j)}{(x_0^2 + a^2 r_j^2 - 2x_0 a r_j \cos\varphi_j)(ar_j - x_0)^2}$$

and the conditions of the theorem show that

$$\sqrt{U_j(\varphi_j)} \leqslant \frac{r_j - x_0}{ar_j - x_0} \quad (j = 1, 2, \dots, n). \tag{3}$$

If, however, $0 \leqslant x_0 \leqslant a$, then from the condition of the theorem and from the equality

$$U_j(\varphi_j) - \left(\frac{r_j + x_0}{ar_j + x_0}\right)^2 = \frac{2x_0 r_j (a-1)(x_0^2 - ar_j^2)(1 + \cos\varphi_j)}{(x_0^2 + a^2 r_j^2 - 2ar_{jx}0 \cos\varphi_j)(ar_j + x_0)^2}$$

it follows that

$$\sqrt{U_j(\varphi_j)} \leqslant \frac{r_j + x_0}{ar_j + x_0} \qquad (j = 1, 2, \dots, n). \tag{4}$$

Moreover, the expression $\frac{r_j - x_0}{ar_j - x_0}\left(\frac{r_j + x_0}{ar_j + x_0}\right)$, as a function of x_0 on the interval $-a \leqslant x_0 \leqslant 0$ $(0 \leqslant x_0 \leqslant a)$, decreases (increases) monotonically. Therefore from (3) and (4) we have

$$\sqrt{U_j(\varphi_j)} \leqslant \frac{r_j + a}{ar_j + a} \qquad (j = 1, 2, \dots, n)$$

or

$$\frac{M(P_n, a)}{M(P_n, 1)} \leqslant a^n \prod_{j=1}^{n} \frac{r_j + a}{a(r_j + 1)} = \prod_{j=1}^{n} \frac{r_j + a}{r_j + 1}. \tag{5}$$

Since

$$\frac{r_j + a}{r_j + 1} \leqslant \frac{a + \sqrt{a} + \gamma}{1 + \sqrt{a} + \gamma} \qquad (j = 1, 2, \dots, n)$$

for $\sqrt{a}+\gamma \leqslant r_j < \infty$, from (5) we find

$$\frac{M(P_n,a)}{M(P_n,1)} \leqslant \left(\frac{a+\sqrt{a}+\gamma}{1+\sqrt{a}+\gamma}\right)^n,$$

as was required to prove.

Let us note that inequality (2) is sharp. One can indicate the general form of the polynomials for which equality is attained in it. In particular, equality in (2) is attained for the polynomial

$$P_n(x) = (x+\sqrt{a}+\gamma)^n.$$

Corollary. *If for the polynomial $P_n(x)$ the inequality*

$$M(P_n,a) > a^{n/2}\left(\frac{1+\sqrt{a}+\gamma/\sqrt{a}}{1+\sqrt{a}+\gamma}\right)^n M(P_n,1), \tag{6}$$

is satisfied, then in the circle $|z| < \sqrt{a}+\gamma$ this polynomial has at least one zero.

Theorem 2. *If $a \geqslant 1,\ r_j \geqslant a+\beta\ (\beta > 0)$, then the inequality*

$$M(P_n,a) \geqslant \left(\frac{\beta}{a+\beta-1}\right)^n M(P_n,1) \tag{7}$$

holds for the polynomial $P_n(x)$.

Proof is analogous to the proof of Theorem 1. Let

$$M(P_n,1) = |P_n(x_0)| \qquad (-1 \leqslant x_0 \leqslant 1).$$

Then we have

$$\frac{M(P_n,1)}{M(P_n,a)} \leqslant \left|\frac{P_n(x_0)}{P_n(ax_0)}\right| = \prod_{j=1}^{n}\sqrt{V_j(\varphi_j)}.$$

It is not difficult to show that

$$\sqrt{V_j(\varphi_j)} \leqslant \begin{cases} \dfrac{r_j+x_0}{r_j+ax_0}, & \text{if } -1 \leqslant x_0 \leqslant 0, \\ \dfrac{r_j-x_0}{r_j-ax_0}, & \text{if } 0 \leqslant x_0 \leqslant 1. \end{cases}$$

Hence, taking into account that the expression

$$\frac{r_j + x_0}{r_j + a x_0} \left(\frac{r_j - x_0}{r_j - a x_0} \right)$$

as a function of x_0 on the interval $-1 \leqslant x_0 \leqslant 0$ ($0 \leqslant x_0 \leqslant 1$) decreases (increases) monotonically, we find

$$\sqrt{V_j(\varphi_j)} \leqslant \frac{r_j - 1}{r_j - a} \quad (j = 1, 2, \dots, n).$$

Consequently,

$$\frac{M(P_n, 1)}{M(P_n, a)} \leqslant \prod_{j=1}^{n} \frac{r_j - 1}{r_j - a}$$

or

$$\frac{M(P_n, 1)}{M(P_n, a)} \leqslant \left(\frac{a + \beta - 1}{\beta} \right)^n$$

when $r_j \geqslant a + \beta$ $(j = 1, 2, \dots, n)$, which was required to be proved.

Corollary. If for a polynomial $P_n(x)$ the inequality

$$M(P_n, a) < \left(\frac{\beta}{a + \beta - 1} \right)^n M(P_n, 1),$$

is satisfied, then in the disk $|z| < a + \beta$ it has at least one zero.

Now consider the rational function

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_0 x^n + a_1 x^{n-1} + \dots + a_n}{b_0 x^m + b_1 x^{m-1} + \dots + b_m}. \tag{8}$$

Denote the zeros of the polynomial $Q_m(x)$ by $t_k = \rho_k e^{i\psi_k}$ $(k = 1, 2, \dots, m)$. The following theorems for rational functions $R(x)$ are proved in a completely analogous way.

Theorem 3. If $a \geqslant 1$, $r_j \geqslant \sqrt{a} + \gamma$ $(\gamma \geqslant 0;\ j = 1, 2, \dots, n)$, $\rho_k \geqslant a + \beta$ $(\beta > 0;\ k = 1, 2, \dots, m)$, then the inequality

$$M(R, a) \leqslant \left(\frac{a + \sqrt{a} + \gamma}{1 + \sqrt{a} + \gamma} \right)^n \left(\frac{a + \beta - 1}{\beta} \right)^m M(R, 1) \tag{9}$$

holds for the rational functions (8).

Theorem 4. If $a \geqslant 1$, $r_j \geqslant a + \beta$ ($\beta > 0$; $j = 1, 2, \dots, n$), $\rho_k \geqslant \sqrt{a} + \gamma$ ($\gamma \geqslant 0$; $k = 1, 2, \dots, m$), then the inequality

$$M(R, a) \geqslant \left(\frac{\beta}{a + \beta - 1}\right)^n \left(\frac{1 + \sqrt{a} + \gamma}{a + \sqrt{a} + \gamma}\right)^m M(R, 1) \tag{10}$$

holds for the rational functions (8).

From Theorems 3 and 4 one can derive the corresponding corollaries on the existence of zeros and poles of the rational functions (8).

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References Cited

1. S. N. Bernstein, *Collected Works*, 1, Publishing House of the Academy of Sciences of the USSR, 1952, p. 11.

2. S. N. Bernstein, *Extremal Properties of Polynomials*, Leningrad–Moscow, 1937.

Note: Figure translations are in progress. See original paper for figures.