ON THE MOTION OF A MONATOMIC RAREFIED GAS IN A HOMOGENEOUSLY EXPANDING SPACE

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Abstract

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ON THE MOTION OF A MONATOMIC RAREFIED GAS IN A HOMOGENEOUSLY EXPANDING SPACE

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In the present paper it is assumed that all equations have been reduced to dimensionless form and that all physical quantities are dimensionless.

In papers ([1,2]), molecular processes occurring in a monatomic gas were considered for the case when, at time t, the macroscopic velocities u, v, w of the particles are distributed in space x, y, z according to the law

$$u = x/t, \qquad v = y/t, \qquad w = z/t \quad (t > 0\text{—expansion},\ t < 0\text{—compression}), \tag{1}$$

and the distribution functions f_i of the peculiar velocities of the molecules do not depend on the spatial coordinates, but depend only on time. The density ρ and temperature T of the gas are the same throughout space and depend only on time. These motions for a continuous medium were obtained by L. I. Sedov ([3]) as a special case of self-similar motions of a gas possessing central symmetry. However, such motions deserve special attention because they possess the property of homogeneity: the velocity fields relative to a coordinate system connected with any moving material particle are identical, and thus the point $x = y = z = 0$ in this sense is in no way distinguished from other points. If the motion is referred to the expanding space with coordinates $\xi = x/t,\ \eta = y/t,\ \zeta = z/t$, then in it these motions give a state of rest, since for any material point $d\xi/dt = d\eta/dt = d\zeta/dt = 0$. In the present paper we shall consider not only states of macroscopic rest of a gas in an expanding space, as in papers ([1,2]), but also a state of macroscopic motion.

The Boltzmann equations for the functions $f_i = f_i(t, r, c_i)$ of the velocity distribution u_i, v_i, w_i of molecules of the i-th species of a monatomic gas have the form ([1])

$$\frac{\partial f_i}{\partial t} + u_i \frac{\partial f_i}{\partial x} + v_i \frac{\partial f_i}{\partial y} + w_i \frac{\partial f_i}{\partial z} = I_i(t, r, c_i). \tag{2}$$

Here r is a vector with components x, y, z; c_i is a vector with components u_i, v_i, w_i; the collision integrals $I_i(t, r, c_i)$ have the form:

$$I_i(t,r,c_i) = \sum_j \iiint \left\{ f_i[t,r,\varphi_{ij}(c_i,c_j,b,e)]\, f_j[t,r,\psi_{ij}(c_i,c_j,b,e)] - f_i(t,r,c_i) f_j(t,r,c_j) \right\} g_{ij} b\, db\, de\, dc_j, \tag{3}$$

where $g_{ij} = |c_i - c_j|$; b is the impact parameter; $dc_j = du_j\, dv_j\, dw_j$; the vectors c_i', c_j' of the velocities of particles i, j after collision are determined by the relations (^{2})

$$c_i' = \varphi_{ij}(c_i, c_j, b, e) = c_i + \frac{m_j}{m_0} \left[(1 - \cos\chi_{ij})(c_j - c_i) - e g_{ij} \sin\chi_{ij} \right]; \tag{4}$$

$$c_j' = \psi_{ij}(c_i, c_j, b, e) = c_j - \frac{m_i}{m_0} \left[(1 + \cos\chi_{ij})(c_i - c_j) - e g_{ij} \sin\chi_{ij} \right]. \tag{5}$$

Here m_i, m_j are the masses of the colliding particles; $m_0 = m_i + m_j$; $\chi_{ij} = \chi_{ij}(b, g_{ij})$ is the angle of rotation of the vector $c_i - c_j$ as a result of the collision; e is a unit vector orthogonal to the vector $c_i - c_j$. The magnitude of the increment of this vector is equal to the quantity $d\varepsilon$ on the right-hand side of equality (3). If between particles i, j there acts a force $P_{ij} = \varkappa_{ij}/d^\nu$, where d is the distance between the particles, $\nu = \text{const} > 0$, $\varkappa_{ij} = \text{const}$, then the function $\chi_{ij}(b, g_{ij})$ has the similarity property (2)

$$\chi_{ij}\left(\alpha^{-2/(\nu-1)} b, \alpha g_{ij}\right) = \chi_{ij}(b, g_{ij}), \tag{6}$$

where α is an arbitrary positive quantity. Relations (4), (5), (6) show that φ_{ij}, ψ_{ij} possess the following addition property:

$$\varphi_{ij}\left(\alpha c_i + q, \alpha c_j + q, \alpha^{-2/(\nu-1)} b, e\right) = \alpha \varphi_{ij}(c_i, c_j, b, e) + q, \tag{7}$$

$$\psi_{ij}\left(\alpha c_i + q, \alpha c_j + q, \alpha^{-2/(\nu-1)} b, e\right) = \alpha \psi_{ij}(c_i, c_j, b, e) + q, \tag{8}$$

where α is an arbitrary positive scalar quantity, and q is an arbitrary vector.

In order to refer the motion to the coordinate system of the expanding space, instead of the variables $t, x, y, z, u_i, v_i, w_i$ we introduce the variables:

$$t; \quad \xi = x/|t|, \quad \eta = y/|t|, \quad \zeta = z/|t|, \quad R = r/|t|,$$

$$U_i = |t|(u_i - x/t), \quad V_i = |t|(v_i - y/t), \quad W_i = |t|(w_i - z/t),$$

$$A_i = |t|(c_i - r/t). \tag{9}$$

Along with the distribution functions $f_i(t, r, c_i)$ satisfying equations (2), we shall consider the distribution functions

$$f_i^* = f_i\left[\tau(t),\, r/|t|,\, |t|(c_i - r/t)\right] = f_i(\tau, R, A_i), \tag{10}$$

where the function $\tau = \tau(t)$ will be defined below. We have

$$f_i^*\left[t, r, \varphi_{ij}(c_i, c_j, b, e)\right] = f_i\left\{\tau, r/|t|,\, |t|\left[\varphi_{ij}(c_i, c_j, b, e) - r/t\right]\right\},$$

$$f_i^*\left[t, r, \psi_{ij}(c_i, c_j, b, e)\right] = f_i\left\{\tau, r/|t|,\, |t|\left[\psi_{ij}(c_i, c_j, b, e) - r/t\right]\right\}.$$

Using the addition properties (7), (8) and putting $\alpha = |t|$, $q = -|t|\, r/t$, we obtain

$$f_i^*\left[t, r, \varphi_{ij}(c_i, c_j, b, e)\right]$$

$$= f_i\left\{\tau, r/|t|, \varphi_{ij}\left[|t|(c_i - r/t),\, |t|(c_j - r/t),\, |t|^{-2/(\nu-1)}b, e\right]\right\}$$

$$= f_i\left[\tau, r/|t|, \varphi_{ij}\left(A_i, A_j, |t|^{-2/(\nu-1)}b, e\right)\right],$$

$$f_i^*\left[t, r, \psi_{ij}(c_i, c_j, b, e)\right] \tag{11}$$

$$= f_i\left\{\tau, r/|t|, \psi_{ij}\left[|t|(c_i - r/t),\, |t|(c_j - r/t),\, |t|^{-2/(\nu-1)}b, e\right]\right\}$$

$$= f_i\left[\tau, r/|t|, \psi_{ij}\left(A_i, A_j, |t|^{-2/(\nu-1)}b, e\right)\right].$$

Denoting by $I_i^*(t, r, c_i)$ the collision integrals for the distribution functions $f_i^*(t, r, c_i)$ and making in equalities (3) the change of variables of integration according to the formulas

$$du_j\, dv_j\, dw_j = \frac{1}{|t|^3}\, dU_j\, dV_j\, dW_j, \qquad b\, db = |t|^{4/(\nu-1)} b_1\, db_1,$$

$$g_{ij} = |c_i - c_j| = \frac{1}{|t|}\, |A_i - A_j|,$$

we obtain

$$I_i^*(t,r,c_i) = |t|^{4\frac{2-\nu}{\nu-1}} I_i\left[\tau(t), R, A_i\right]. \tag{12}$$

Requiring that the functions $f_i^*(t,r,c_i)$ satisfy equations (2), and changing in the left-hand sides of these equalities from the variables $t, x, y, z, u_i, v_i, w_i$ to the variables $t, \xi, \eta, \zeta, U_i, V_i, W_i$, we obtain

$$\frac{\partial f_i^*}{\partial t} + \frac{1}{t^2}\left(U_i \frac{\partial f_i^*}{\partial \xi} + V_i \frac{\partial f_i^*}{\partial \eta} + W_i \frac{\partial f_i^*}{\partial \zeta}\right) = I_i^*(t,r,c_i). \tag{13}$$

Making the substitutions according to formulas (11), (12), we obtain:

$$\frac{d\tau}{dt}\,\frac{\partial f_i(\tau,R,A_i)}{\partial \tau} + \frac{1}{t^2}\left[U_i \frac{\partial f_i(\tau,R,A_i)}{\partial \xi} + V_i \frac{\partial f_i(\tau,R,A_i)}{\partial \eta} + W_i \frac{\partial f_i(\tau,R,A_i)}{\partial \zeta}\right] = |t|^{4\frac{2-\nu}{\nu-1}} I_i[\tau,R,A_i]. \tag{14}$$

Since the functions $f_i(t,r,c_i)$ satisfy equations (2), we have

$$\frac{\partial f_i(\tau,R,A_i)}{\partial \tau} + U_i \frac{\partial f_i(\tau,R,A_i)}{\partial \xi} + V_i \frac{\partial f_i(\tau,R,A_i)}{\partial \eta} + W_i \frac{\partial f_i(\tau,R,A_i)}{\partial \zeta} = I_i(\tau,R,A_i).$$

Therefore the equalities (14) will be satisfied if one sets

$$\nu = 3; \qquad \frac{d\tau}{dt} = \frac{1}{t^2}; \qquad \tau = \beta - \frac{1}{t}, \qquad \beta = \text{const}. \tag{15}$$

Thus, for $\nu = 3$, to any solution $f_i = f_i(t,r,c_i)$ of the Boltzmann equations there corresponds a solution $f_i^*(t,r,c_i) = f_i[\beta - 1/t,\ r/|t|,\ |t|(c_i - r/t)]$ of the same equations. The distribution of relative velocities $c_i - r/t$ for f_i^* and the distribution of absolute velocities for f_i are similar; moreover, for f_i^* as $t \to \infty$ (expansion), over the entire infinite interval of time there is realized a state similar to that which for f_i is realized over a finite interval of time.

For compression flows $-\infty < t < 0$, as $t \to -0$, for f_i^* there is realized a distribution of relative velocities similar to that which is realized for f_i as $t \to \infty$ for the absolute velocities.

The general similarity law derived above for a monatomic gas with interaction law $\nu = 3$ must be valid, in particular, for motions that can be characterized by the Navier–Stokes equations of motion of a viscous heat-conducting gas, and also by the equations of an ideal (nonviscous and non-heat-conducting) gas. It is known that for any monatomic gas the adiabatic exponent is $\gamma = 5/3$, and the coefficients of heat conductivity and viscosity, for the case of a power-law interaction, are proportional to the quantity T^S, where T is temperature, $S = 1/2 + 2/(\nu - 1)$ (see ([4]), Ch. 10, § 3). For the case considered by us, $\nu = 3$, $S = 3/2$.

The system of equations (1, 2), (15,5), (49,5) of book ([5]), in which for a monatomic gas $\zeta = 0$ (see ([6]), Ch. 7, § 6, p. 399; Ch. 8, § 1, p. 414), with the same notation and symbolism as in ([5]), and also under the assumptions $\eta = C_\eta T^{3/2}$, $\varkappa = C_\varkappa T^{3/2}$, where $C_\eta, C_\varkappa$ are constants, will take the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0; \tag{16}$$

$$\rho \left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} C_\eta \left\{ T^{3/2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) \right\}; \tag{17}$$

$$\rho T \left(\frac{\partial S}{\partial t} + v \nabla S \right) = C_\varkappa \operatorname{div} \left(T^{3/2} \nabla T \right) + \frac{C_\eta}{2} T^{3/2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right)^2. \tag{18}$$

Let us make a change of variables according to the formulas

$$v_i = \frac{\bar{v}_i + x_i}{t}, \qquad p = t^{-5} \bar{p}, \qquad \rho = t^{-3} \bar{\rho}, \qquad T = t^{-2} \bar{T},$$

$$\tau = \beta - \frac{1}{t}, \qquad \xi_i = \frac{x_i}{t}, \qquad \beta = \text{const}.$$

In these variables, equations (16), (17) have exactly the same form, except that in them t must be replaced by τ, x_i by ξ_i, v_i by $\bar{v}_i$, p by $\bar{p}$, ρ by $\bar{\rho}$, T by $\bar{T}$. Hence it follows that if the functions $v_i(t, x_1, x_2, x_3)$ $(i = 1, 2, 3)$, $\rho(t, x_1, x_2, x_3)$, $p(t, x_1, x_2, x_3)$, $T(t, x_1, x_2, x_3)$ satisfy equations (16), (17), (18), then the functions

$$v_i^*(t, x_1, x_2, x_3) = \frac{1}{t} v_i \left(\beta - \frac{1}{t}, \frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t} \right) + \frac{x_i}{t}; \tag{19}$$

$$\rho^*(t, x_1, x_2, x_3) = t^{-3} \rho \left(\beta - \frac{1}{t}, \frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t} \right); \tag{20}$$

$$p^*(t, x_1, x_2, x_3) = t^{-5} p \left(\beta - \frac{1}{t}, \frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t} \right); \tag{21}$$

$$T^*(t, x_1, x_2, x_3) = t^{-2} T\left(\beta - \frac{1}{t}, \frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t}\right). \tag{22}$$

The regularities obtained for a viscous gas are valid not for an arbitrary monatomic gas, but for a particular law of interaction between molecules (a particular law of dependence of the coefficients of friction and thermal conductivity on T). However, if friction and heat transfer are absent, then expressions (19), (20), (21), (22) satisfy the equations of motion for any monatomic gas, since for this it is only necessary that the adiabatic exponent γ of the gas be equal to 5/3. This is easily verified also by direct substitution of the solutions (19), (20), (21), (22) into the equations of motion. In motions characterized by the quantities with asterisks, as $t \to \infty$, processes occur over the entire infinite time interval similar to those which, in the original motion, are realized over a quite definite finite interval of time. Relations (19), (20), (21), (22) make it possible, in particular, starting from exact solutions of the equations of dynamics of an ideal monatomic gas already obtained by various authors, to obtain the same number of new exact solutions.

The author' s paper (7) is a continuation of the present work.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.