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ON THE CONVERGENCE OF GRADIENT METHODS FOR FINDING A LOCAL CONSTRAINED MINIMUM OF A NONLINEAR FUNCTIONAL UNDER LINEAR CONDITIONS IN HILBERT SPACE

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Abstract

Full Text

MATHEMATICS

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ON THE CONVERGENCE OF GRADIENT METHODS FOR FINDING A LOCAL CONSTRAINED MINIMUM OF A NONLINEAR FUNCTIONAL UNDER LINEAR CONDITIONS IN HILBERT SPACE

(Presented by Academician A. A. Dorodnitsyn, 23 I 1963)

Let $f(x)$ be a functional defined in some convex domain D of a real Hilbert space H, and let π be a plane in H specified by the equations

$$(c^i, x) = b_i, \qquad i = 1, 2, \dots, m, \tag{1}$$

and having a nonempty intersection with the domain D. We assume that the elements c^i are orthonormal. Denote

$$f'(x)y = (\nabla f(x), y), \qquad f''(x)yz = (H_f(x)y, z). \tag{2}$$

If $x_0 \in D \cap \pi$ is an initial approximation to a constrained local minimum of the functional $f(x)$ under conditions (1), then it is natural to seek the point x^* at which this minimum is attained by moving near the curve of "steepest descent," which is defined by the differential equation

$$x'(t) = -\nabla f(x(t)) + \sum_{i=1}^{m} (\nabla f(x(t)), c^i)c^i \tag{3}$$

and the initial condition $x(0) = x_0$.

Theorem 1. *If the functional $f(x)$ is twice continuously differentiable in the convex domain $D \cap \pi$, $x_0 \in \pi$, the closed sphere $S[x_0, r] \cap D$, where $r \geq MG(x_0)$, and*

$$G^2(x) = \|\nabla f(x)\|^2 - \sum_{i=2}^{m} (\nabla f(x), c^i)^2, \tag{4}$$

$$\nabla f(x)$$

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satisfies the Lipschitz condition in the sphere $S[x_0, r] \cap \pi$, *and for any* $x \in D \cap \pi$ *and* $y \in H$ *the condition*

$$(H_f(x)y, y) \geq \frac{1}{M}\|y\|^2 \qquad (M > 0) \tag{5}$$

is fulfilled, and $x(t)$ *is a solution of equation (3), then for* $t \geq 0$

$$x(t) \in S[x_0, r] \cap \pi, \tag{6}$$

the limits exist

$$\lim_{t\to\infty} x(t) = x^*, \qquad \lim_{t\to+\infty} f(x(t)) = c, \tag{7}$$

and for $t > 0$ *the estimates hold*

$$\|x(t) - x^*\| \leq MG(x_0)\exp\left(-\frac{t}{M}\right), \tag{8}$$

$$0 \leq f(x(t)) - c \leq \frac{M}{2}G^2(x_0)\exp\left(-\frac{2t}{M}\right), \tag{9}$$

and for all $x \in D \cap \pi$

$$f(x) \geq c + \frac{\|x - x^*\|}{2M}. \tag{10}$$

In the case of an unconstrained minimum ($m = 0$), Theorem 1 reduces to the main theorem in the work ([1]) of P. S. Rosenbloom.

The connection established by Theorem 1 between the solution of differential equation (3) and the point x^* of the conditional minimum, with linear conditions, of the functional $f(x)$ makes it possible to construct and investigate for convergence various methods of approximate determination of the point x^*, based on methods of approximate solution of differential equation (3).

We introduce the notation: $p(\delta)$ is a polynomial of degree $s - 2$ with positive coefficients,

$$q(h, \delta) = (m+1)Ah^s p(\delta) + \exp\left(-\frac{h}{M}\right), \tag{11}$$

$$r(h, \delta) = \frac{h^s p(\delta) + M\left[1 - \exp\left(-\frac{h}{M}\right)\right]}{1 - q(h, \delta)}\,\delta, \tag{12}$$

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and the sequence $\{\delta_n\}$ is defined by the formula

$$\delta_{n+1} = q(h, \delta_n)\delta_n \qquad (n = 0, 1, ...). \tag{13}$$

For one-step methods we have the following result:

Theorem 2. Let the functional $f(x)$ be twice continuously differentiable in the convex domain $D \cap \pi$, $x_0 \in \pi$, and let the closed sphere $S[x_0, r(h, \delta_0)] \subset D$, where $\delta_0 \geqslant G(x_0)$; suppose that $\nabla f(x)$ satisfies, in the ball $S[x_0, r(h, \delta_0)] \cap \pi$, the Lipschitz condition with constant A, and that in $D \cap \pi$ condition (5) is fulfilled. Further, let h^* be the positive root of the equation

$$q(h, \delta_0) = 1, \tag{14}$$

let h be some number, $0 < h < h^*$, and let the sequence of elements of the plane π be constructed by the recurrence formula

$$x_{n+1} = F(x_n, h) \qquad (n = 0, 1, ...), \tag{15}$$

where for all n the condition

$$\|F(x_n, h) - x_n(h)\| \leqslant h^s p(\|G(x_n)\|)\|G(x_n)\|, \qquad s \geqslant 2,$$

is satisfied, where $x_n(t)$ is the solution of differential equation (3) under the initial condition $x_n(0) = x_n$.

Then $x_n \in S[x_0, r(h, \delta_0)] \cap \pi$, and the sequence $\{x_n\}$ converges to the unique conditional minimum x^* of the functional $f(x)$ in the domain D under conditions (1), with rate

$$\|x_{n+1} - x^*\| \leqslant \left[h^s p(\delta_n) + M \exp\left(-\frac{h}{M}\right)\right] \delta_n. \tag{16}$$

As examples of the application of Theorem 2, we give two theorems corresponding to the approximate solution of equation (3) by Euler' s method and by the improved Euler–Cauchy method.

Theorem 3. If the functional $f(x)$ is twice continuously differentiable in the convex domain $D \cap \pi$, $x_0 \in \pi$, the closed sphere $S[x_0, r] \subset D$, where

$$r \geqslant \frac{2M[1 - \exp(-h/M)] + (m+1)Ah^2}{2[1 - \exp(-h/M)] - (m+1)^2 A^2 h^2}\, G(x_0), \tag{17}$$

in the ball $S[x_0, r] \cap \pi$ condition (5) and the condition

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$$\|H_f(x)\| \leqslant A \tag{18}$$

are fulfilled, and h satisfies the inequality

$$0 < h < \frac{2M}{1+(m+1)^2A^2M^2}, \tag{19}$$

then the sequence

$$x_{n+1} = x_n - h\left[\nabla f(x_n) - \sum_{i=1}^{m}(\nabla f(x_n), c^i)c^i\right] \tag{20}$$

converges to the unique conditional minimum x^* of the functional $f(x)$ in the domain D under conditions (1) with the rate

$$\|x_{n+1}-x^*\| \leq \left[\frac{m+1}{2}Ah^2 + M\exp\left(-\frac{h}{M}\right)\right]\left[\frac{(m+1)^2}{2}A^2h^2 + \exp\left(-\frac{h}{M}\right)\right]^n G(x_0). \tag{21}$$

Theorem 4. Let the functional $f(x)$ be twice continuously differentiable in the convex domain $D \cap \pi$, $x_0 \in \pi$, and let the closed sphere $S[x_0, r] \subset D$, where

$$r \geq \frac{12M[1-\exp(-h/M)] + h^3\left[5(m+1)ABG(x_0) + 2(m+1)^2A^2\right]}{12[1-\exp(-h/M)] - (m+1)Ah^3\left[5(m+1)ABG(x_0) + 2(m+1)^2A^2\right]}G(x_0), \tag{22}$$

and suppose that in the ball $S[x_0, r] \cap \pi$ conditions (5) and (18) are satisfied, and $f(x)$ is three times continuously differentiable in the ball $S[x_0, r + hG(x_0)] \cap \pi$, and moreover in this ball

$$\|H_f'(x)\| \leq B. \tag{23}$$

If, in addition, h satisfies the inequality

$$0 < h < \frac{2\sqrt{3}\,M}{2.118117 + \sqrt{5(m+1)ABM^3G(x_0) + 2(m+1)^3A^3M^3 - 2}}, \tag{24}$$

then the sequence

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$$x_{n+1} = x_n - \frac{h}{2}\left[\nabla f(x_n) + \nabla f(y_n) - \sum_{i=1}^{m}(\nabla f(x_n) + \nabla f(y_n), c^i)c^i\right], \qquad (25)$$

where

$$y_n = x_n - h\left[\nabla f(x_n) - \sum_{i=1}^{m}(\nabla f(x_n), c^i)c^i\right], \qquad (26)$$

converges to the unique conditional minimum x^* of the functional $f(x)$ in the domain D under conditions (1) with the rate

$$\|x_{n+1} - x^*\| \leq \left[h^3\left(\frac{5}{12}B\delta_n + \frac{(m+1)^2}{6}A^2\right) + M\exp\left(-\frac{h}{M}\right)\right]\delta_n, \qquad (27)$$

where $\delta_0 \geq G(x_0)$ and

$$\delta_k = \frac{5}{12}(m+1)h^3AB\delta_{k-1}^2 + \left[\frac{(m+1)^2}{6}h^3A^3 + \exp\left(-\frac{h}{M}\right)\right]\delta_{k-1}. \qquad (28)$$

Analogous methods and theorems on their convergence can be obtained by starting from Runge–Kutta methods of higher orders. If in Theorems 2–4 one takes $m = 0$ $\left(\sum_{i=1}^{m} = 0\right)$, then one obtains theorems on the convergence of methods for the approximate finding of unconditional local minima of the functional $f(x)$.

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CITED LITERATURE

1. P. S. Rosenbloom, Numerical Analysis, Proc. Symposia in Appl. Math., **6**, N. Y., 1956.

Note: Figure translations are in progress. See original paper for figures.

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