ON SOME PROPERTIES OF ARITHMETIC OPERATIONS ON DUPLEXES

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Abstract

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MATHEMATICS

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ON SOME PROPERTIES OF ARITHMETIC OPERATIONS ON DUPLEXES

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1. In the present note, all terms and notations not specially explained are understood in the same way as in ([1,2]); all statements are understood in the sense of the constructive interpretation (see ([3])).

We shall say that a natural number n is an **index of self-convergence** of a sequence of rational numbers *g for a natural number k, if for all m, l such that $m, l \geq n$,

$$|g(m) - g(l)| < 2^{-k}.$$

A sequence of natural numbers f is called a **regulator of self-convergence** ** of the sequence g, if for every k the natural number $f(k)$ is an index of self-convergence of the sequence g for k.

Every rational number and every word of the form $P \diamond Q$, where P is a notation of a sequence of rational numbers, and Q is a notation of its regulator of self-convergence, is called a **duplex** ([2]). If H is a duplex of the form $P \diamond Q$, then by $\underline{H}$ and $\overline{H}$ are denoted, respectively, the sequence of rational numbers whose notation is P, and the sequence of natural numbers whose notation is Q. If, however, H is a rational number, then $\underline{H}$ and $\overline{H}$ denote, respectively, the sequence of rational numbers and the sequence of natural numbers such that, for every natural k,

$$\underline{H}(k) = H, \qquad \overline{H}(k) = 0.$$

The algorithm $\underline{H}$ is called the **basis** of the duplex H.

In the subsequent exposition the letters x, y, z, u are variables for duplexes; the letters k, l, m, n are variables for natural numbers; $\square$ is a separating sign.

We shall say that a natural number n is a **minimal index of self-convergence** *** of a sequence g for a natural number k, if n is an index of self-convergence of g for k and is majorized by every other index of self-convergence of g for k. We shall say that a sequence of natural numbers f is a **minimal regulator of**

convergence of the sequence g, if for every k the number $f(k)$ is an MISC of g for k.

Theorem 1. *There is no algorithm ψ which transforms every duplex into a natural number and such that the condition "$\overline{x}(\psi(x)) - 1$ is an index of self-convergence of the sequence $\underline{x}$ for $\psi(x)$," having the duplex variable x as its parameter, is algorithmically verifiable for admissible values of the parameter.*

Corollary. *There is no algorithm which constructs, for an arbitrary duplex, a minimal regulator of convergence of its basis.*

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* Sequences of rational numbers are normal algorithms ([1]) in a certain standard alphabet ([2]), transforming every natural number into a rational one. In what follows, details connected with alphabets are omitted.

** Instead of the words "regulator of self-convergence" we shall write RC.

*** Instead of the words "minimal index of self-convergence" we shall write MISC.

2. Let us construct algorithms V, N, R so that for any x, n, k

$$V(x\Box n\Box k) \simeq \begin{cases} x\nabla x, & \text{if } x \text{ is a rational number,} \\ \underline{x}(\overline{x}(n)+k) - 2^{-n}\nabla \underline{x}(\overline{x}(n)+k) + 2^{-n}, & \text{otherwise}^*; \end{cases}$$

$$N(x\Box n) \simeq \max(n+1\Box\overline{x}(0)\Box\overline{x}(1)\Box\ldots\Box\overline{x}(n+1));$$

$$R(x\Box n) \simeq \bigcap_{m=0}^{n+1} \bigcap_{l=0}^{N(x\Box n)-\overline{x}(m)} V(x\Box m\Box l).$$

Lemma. One can construct an algorithm U such that, whatever x, n, k may be:

1) $U(x\Box n\Box k) \neq \Lambda$ if and only if $\overline{x}(n) - k$ is not an indicator of convergence in itself of the sequence $\underline{x}$ for the number n;

2) condition **

$$U(x\Box n\Box k-1) = \Lambda \ \&\ \neg!\, U(x\Box n\Box k) \qquad (1)$$

is satisfied if and only if one can construct a rational number a such that:

a) $x = a$; b) $a \in R(x\Box n + l)$ for all l; c) $|\Gamma - a| = 2^{-n}$;

b) whatever m may be, if $m > \overline{x}(n) - k$, then $|\Gamma - \underline{x}(m)| < 2^{-n}$ and $|a - \underline{x}(m)| \leq 2^{-n}$;

c) whatever m may be, if $\overline{x}(n) - k < m < \overline{x}(n)$, then $|a - \underline{x}(m)| = 2^{-n}$; here Γ denotes the expression $\underline{x}(\overline{x}(n) - k)$.

On the basis of this lemma the following is proved.

Theorem 2. Whatever the duplex x may be, if $\neg\exists nk\,\mathfrak{A} \vee \exists nk\,\mathfrak{A}$, where $\mathfrak{A}$ denotes formula (1), then one can construct a minimal CS of the sequence $\underline{x}$.

From Theorem 2 it follows immediately:

Theorem 3. It is impossible to construct a duplex for whose base a minimal CS would be impossible.

In connection with this theorem we note that for every duplex one can construct a duplex equal to it (in the sense of equality of duplexes) with a minimal CS.

3. In the subsequent exposition the sign $\odot$ will symbolize any of the signs $+, -, \cdot, :, \Lambda^*$. The expression $\odot$ will denote the corresponding arithmetic operation on duplexes in the form in which it is defined in ([2]) (Λ^* denotes the operation of raising the modulus of a duplex to a duplex power). The definition of each of these operations can be varied by changing the manner of obtaining, from the initial data, a CS of the result of the operation (while preserving the law of computing the base of the result of the operation).

Let $F^{\odot}$ be one of such variants of the operation $\odot$ (in this paragraph Λ^* is excluded from the number of admissible values for $\odot$). Let 2^{-n} be the precision with which we wish to obtain a rational approximation to the duplex $F^{\odot}(x\square y)$, where x and y are given duplexes. Let us denote the expression $F^{\odot}(x\square y)$ by the letter ω. Compute $\overline{\omega}(n)$. This natural number is always not less than the MCS of the sequence $\underline{\omega}$ for the number n. It has the following property: whatever l may be, if $l \geq \overline{\omega}(n)$, then the application of the standard operation $\odot$ (coinciding on rational numbers with $F^{\odot}$) to the rational numbers $\underline{x}(l)$ and $\underline{y}(l)$ gives a rational approximation to the duplex ω with the desired precision. The number $\overline{\omega}(n)$ will differ from the MCS of the sequence $\underline{\omega}$

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* Words of the form $S\vee T$ and $S\nabla T$, where S and T are duplexes, are respectively the segment and the interval with endpoints S and T.

** An expression of the form $!f(P)$ is read as: “the process of applying the algorithm f to the word P terminates.”

for n to a greater or lesser degree depending on which variant $F^{\odot}$ of the operation $\odot$ has been chosen. For some initial data x, y and some n it may turn out that the number $\overline{\omega}(n)$ coincides with the MIS of the sequence $\underline{\omega}$ for n.

Below we consider the question of how variation of the operation $\odot$ affects the set of those pairs of duplexes $x\square y$ for which, for a given n, the indicated coincidence occurs.

Let $\mathfrak{P}$ be some subset (a constructive concept of a set is meant ([3])) of the set of all duplexes. We shall say that an algorithm F, which transforms a pair of duplexes from $\mathfrak{P}$ into a duplex, is an **arithmetic operation of type** $\odot$, if for any x, y, z, u from $\mathfrak{P}$ the following conditions are satisfied: 1) if $x = z$, $y = u$, and $!F(x\Box y)$, then $!F(z\Box u)$ and $F(x\Box y) = F(z\Box u)$ ($x = y$ means that $\underline{x}(k) = \underline{y}(k)$ and $\overline{x}(k) = \overline{y}(k)$ for all k); 2) F is applicable to the pair $x\Box y$ if and only if the operation $\odot$ is applicable to the same pair; 3) if $!F(x\Box y)$, then

$$\underline{F(x\Box y)}(k) = \underline{x \odot y}(k)$$

for any k.

Thus, an operation F of type $\odot$ can differ from the standard operation $\odot$ only in the method of obtaining the RS of the result. In order to exclude uninteresting cases from consideration, we shall assume that arithmetic operations satisfy the condition: 4)

$$\overline{F(x\Box y)}(k) \leqslant \overline{x \odot y}(k)$$

for any x and y from $\mathfrak{P}$ and any k.

Below the letter Ω symbolizes either of two sets: the set of all duplexes and the set of duplexes with minimal RS. The letters $F^{\odot}, F_1^{\odot}, F_2^{\odot}$ will be used as variables for arithmetic operations of type $\odot$ on the set Ω.

Theorem 4. *One can construct such duplexes x and y from Ω that, for any arithmetic operation $F^{\odot}$ on Ω, first, $!F^{\odot}(x\Box y)$ and, second, for no k is the number*

$$\overline{F^{\odot}(x\Box y)}(k)$$

the MIS of the base of the duplex $F^{\odot}(x\Box y)$ for k.

Construct duplexes $A^{\odot}$ and $B^{\odot}$ so that for all k:

$$\underline{A^{+}}(k) = 5 \cdot 2^{-(k+2)}, \qquad \overline{A^{+}}(k) = k + 1, \qquad \underline{B^{+}}(k) = 3 \cdot 2^{-(k+2)},$$

$$\overline{B^{+}}(k) = k; \qquad A^{-} \simeq A^{+},$$

$$\underline{B^{-}}(k) = -\underline{B^{+}}(k), \qquad \overline{B^{-}}(k) = k; \qquad \underline{A^{\cdot}}(k) = 3 \cdot 2^{-k}, \qquad \overline{A^{\cdot}}(k) = k + 2,$$

$$B^{\cdot} \simeq 1/3; \qquad A^{\div} \simeq A^{\cdot},$$

$$B^{\div} \simeq 3; \qquad \underline{A^{\Lambda *}}(k) = \left(1 - 2^{-(k+1)}\right)^2, \qquad \overline{A^{\Lambda *}}(k) = k, \qquad B^{\Lambda *} \simeq 1/2.$$

The duplexes $A^{\odot}$ and $B^{\odot}$ are duplexes with minimal RS. They will be the required ones: any operation $F^{\odot}$ transforms the pair $A^{\odot}\Box B^{\odot}$ into a duplex whose RS, for not a single k, coincides with the MIS of the base of this duplex for k.

Theorem 5. *For any arithmetic operation $F_1^{\odot}$ and natural number k, one can construct an arithmetic operation $F_2^{\odot}$ and duplexes x and y from Ω such that $!F_1^{\odot}(x\Box y)$,*

$$\overline{F_1^{\odot}(x\Box y)}(k)$$

is not the MIS of the base of the duplex $F_1^{\odot}(x\Box y)$ for k, while $F_2^{\odot}(x\Box y)$ is a duplex with minimal RS.

Corollary. It is impossible for there to be an arithmetic operation of type $\odot$ such that the complete preimage (in the set of pairs of duplexes) of the set of duplexes with minimal RC would contain the complete preimage of this set under every other operation of the same type $\odot$.

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Note: Figure translations are in progress. See original paper for figures.