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Abstract

Full Text

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VARIATIONAL PROBLEMS OF OPTIMIZATION OF CONTROL PROCESSES WITH FUNCTIONALS DEPENDING ON INTERMEDIATE VALUES OF THE COORDINATES

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In the present paper we give the results obtained by the author in the problem indicated in the title. Let us begin with its formulation.

Consider the system of differential equations [3]

$$g_s = \dot{x}_s - f_s(x_1, \dots, x_n,\ u_1, \dots, u_m,\ t) = 0 \quad (s = 1, \dots, n) \tag{1}$$

and the terminal relations

$$\psi_k = \psi_k(u_1, \dots, u_m,\ t) = 0 \quad (k = 1, \dots, r < m), \tag{2}$$

in which x_s are coordinates, and u_k are control parameters. We shall assume that they describe the behavior of some mechanical system.

We shall assume that in the $(n+m)$-dimensional space $x_1, \dots, x_n,\ u_1, \dots, u_m$ there is an open domain of their admissible variations. Further, let there be an interval $t_0 \le t \le t_{q+1}$ and, within it, points $t = t_i$ $(i = 1, \dots, q)$, at which the relations

$$\varphi_l = \varphi_l[x(t_0), t_0, x(t_1), t_1, \dots, x(t_q), t_q, x(t_{q+1}), t_{q+1}] = 0, \tag{3}$$

$$l = 1, \dots, p \le (q+2)(n+1) - 1.$$

hold.

In them, for brevity of notation, the set $x_1(t), \dots, x_n(t)$ is denoted by $x(t)$. Let us pose the following variational problem.

Among continuous functions $x_s(t)$ $(s = 1, \dots, n)$ with piecewise-continuous derivatives $\dot{x}_s(t)$, and among piecewise-continuous controls $u_k(t)$ $(k = 1, \dots, r)$, subject to the equations (1) and (2) on the interval $t_0 \le t \le t_{q+1}$, find those which give the functional

$$J = g[x(t_0), t_0, x(t_1), t_1, \dots, x(t_q), t_q, x(t_{q+1}), t_{q+1}] +$$

$$+ \int_{t_0}^{t_{q+1}} f_0(x_1, \dots, x_n,\ u_1, \dots, u_m,\ t)\, dt \qquad (4)$$

a minimum (or maximum) value under the condition that the quantities $x_s(t_i)$ $(s = 1, \dots, n;\ i = 0, \dots, q+1)$ are connected by the relations (3).

The functional J contains the same values $x_s(t_i)$ as do the equalities (3). The relations (2), just as was done in other, simpler problems [4], may be used for passing from closed to open domains of admissible variations of the control parameters u_k. It is assumed that the functions f_s $(s = 0, 1, \dots, n)$, ψ_k $(k = 1, \dots, r)$, φ_l $(l = 1, \dots, p)$, and g are continuous

and have continuous partial derivatives with respect to their arguments up to the third order inclusive ([1]).

Many problems of optimizing control processes lead to such a variational formulation. In particular, it is encountered in finding laws of variation of the control parameters that impart a maximum or minimum to extremal values of coordinates or of some of their functions.

By suitably modifying the corresponding arguments described, for example, in ([1]), one can prove the fundamental lemma on inclusion in a family of comparison curves. The additional conditions will include the requirement that the rank of the matrix

$$\frac{\partial \Psi}{\partial u} = \left\| \frac{\partial \Psi_k}{\partial u_\rho} \right\|$$

be equal to r, and the requirement that the “conditions of non-tangency”

$$\frac{\partial \varphi_l}{\partial t_i} + \sum_{\alpha=1}^{n} \frac{\partial \varphi_l}{\partial x_\alpha(t_i)} \dot{x}_\alpha(t_i) \ne 0 \qquad (l = 1, \dots, p;\ i = 1, \dots, q).$$

be satisfied.

They must hold on the curve included in the family of comparison curves.

Having considered in the space $(x_1, \dots, x_n, u_1, \dots, u_m)$ a curve C that imparts a minimum (or maximum) to the functional J, and having constructed the corresponding family of comparison curves, one can establish the first necessary condition for a minimum (or maximum) of the functional J. It may be formulated as follows.

On the curve C that imparts a minimum (or maximum) to the functional J, the equality

$$\Delta I = 0, \tag{5}$$

must be satisfied, where

$$I = \varphi + \int_{t_0}^{t_{q+1}} \mathcal{L}\, dt; \tag{6}$$

$$\varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l; \tag{7}$$

$$\mathcal{L} = f_0 + \sum_{s=1}^{n} \lambda_s g_s - \sum_{k=1}^{r} \mu_k \psi_k = \sum_{s=1}^{n} \lambda_s \dot{x}_s - H; \tag{8}$$

$$H = H_\lambda + H_\mu = \sum_{s=0}^{n} \lambda_s f_s + \sum_{k=1}^{r} \mu_k \psi_k \qquad (\lambda_0 = -1) \tag{9}$$

and by ΔI is denoted the total variation of the functional I. The multipliers $\lambda_s(t)$ and $\mu_k(t)$ may have discontinuities of the first kind in the interval $t_0 \le t \le t_{q+1}$ at the points $t = t_i$ and at the values of t corresponding to corner points of the curve.

Expanding relation (5), we obtain the differential equations

$$\dot{\lambda}_s + \frac{\partial H}{\partial x_s} = 0 \qquad (s = 1, \dots, n), \tag{10}$$

the equalities

$$\frac{\partial H}{\partial u_k} = 0 \qquad (k = 1, \dots, m), \tag{11}$$

BOUNDARY CONDITIONS

$$\lambda_s(t_0) - \frac{\partial \varphi}{\partial x_s(t_0)} = 0, \qquad \lambda_s(t_{q+1}) + \frac{\partial \varphi}{\partial x_s(t_{q+1})} = 0 \quad (s = 1, \dots, n),$$

$$(H_\lambda)_{t_0} + \frac{\partial \varphi}{\partial t_0} = 0, \qquad (H_\lambda)_{t_{q+1}} - \frac{\partial \varphi}{\partial t_{q+1}} = 0 \tag{12}$$

and the Erdmann–Weierstrass conditions

$$\lambda_s^-(t_i) - \lambda_s^+(t_i) + \frac{\partial \varphi}{\partial x_s(t_i)} = 0, \quad (s = 1, \dots, n;\ i = 1, \dots, q),$$

$$\frac{\partial \varphi}{\partial t_i} - (H_\lambda^-)_{t_i} + (H_\lambda^+)_{t_i} = 0 \quad (i = 1, \dots, q). \tag{13}$$

Adding to them equations (1) and (2), the relations (3), and the continuity conditions for the coordinates

$$x_s^-(t_i) = x_s^+(t_i) \quad (s = 1, \dots, n;\ i = 1, \dots, q),$$

we arrive at a set of equations and relations determining curves that satisfy the stationarity condition.

Unlike the constrained problems of Lagrange, Mayer, and Bolza in the calculus of variations ([1,2]), here one has to deal with the construction of a system of Lagrange multipliers $\lambda_1(t), \dots, \lambda_n(t)$ with discontinuities of the first kind. The function H_λ may also have discontinuities of the first kind. In the case when φ_l does not depend explicitly on the quantity $x_s(t_i)$, the multiplier $\lambda_s(t_i)$ will be continuous at the point $t = t_i$. If all t_i $(i = 1, \dots, q)$ do not enter the relations (3), and the functions f_s and ψ_k do not contain the time t explicitly, then the first integral $H_\lambda = \text{const}$ holds.

An admissible curve C that gives a minimum to the functional J also satisfies the second necessary condition for a minimum—the Weierstrass condition. In its formulation, the Weierstrass function is used, equal to

$$E = \mathcal{L}(x, \dot{X}, U, \lambda, \mu, t) - \mathcal{L}(x, \dot{x}, u, \lambda, \mu, t) - \sum_{r=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{x}_s}(\dot{X}_s - \dot{x}_s). \tag{14}$$

Here x_s and u_k denote the functions corresponding to the curve that gives a minimum to the functional, while X_s and U_k denote arbitrary admissible functions.

An admissible curve C satisfies the necessary Weierstrass condition for a strong minimum of the functional J if, for it, the stationarity condition holds with multipliers $\lambda_s(t)$ and $\mu_k(t)$, and with these multipliers the inequality

$$E \geq 0 \tag{15}$$

is satisfied for all possible admissible quantities $\dot{X}_s,\ U_k \neq \dot{x}_s,\ u_k$, connected by equations (1) and (2).

In connection with the discontinuities of the multipliers $\lambda_s(t)$, the Weierstrass condition at the points $t = t_i$ must be checked for the left and right limits of the function E. Substituting expression (8) into the function E, we arrive at the following form of inequality (15):

$$H_\lambda(x, u, \lambda, t) \geq H_\lambda(x, U, \lambda, t). \tag{16}$$

It is usually used in solving problems of optimization of control processes. Inequality (16) must be satisfied for any admissible quantities $U_k \neq u_k$, where u_k corresponds to the curve giving a minimum to the functional J.

Passing to comparison curves close to the curve C, one can, with the aid of the necessary Weierstrass condition, obtain the necessary Clebsch condition for a weak minimum of the functional. The necessary Jacobi condition for the positivity of the second variation of the functional J, and sufficient conditions for a minimum of this functional, are established with considerably greater difficulty. They are not given here, since they have not yet found proper application in solving problems of optimization of control processes.

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Note: Figure translations are in progress. See original paper for figures.