ON THE QUESTION OF THE INDEX OF A SYSTEM OF SINGULAR EQUATIONS

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Abstract

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MATHEMATICS

1°. Let Γ be a manifold without boundary of some dimension $m \geq 2$. We shall assume that it consists of a finite number n of connected Lyapunov manifolds $\Gamma_1, \Gamma_2, \dots, \Gamma_n$, which pairwise have no common points and are all compact, except, possibly, for one, which then coincides with the m-dimensional Euclidean space E_m. Consider the system of singular integral equations

$$\sum_{h=1}^{s} \left\{ a_{jh}(x)u_h(x) + \int_{\Gamma} K_{jh}(x,y)u_h(y)\,d\Gamma_y + T_{jh}u_h \right\} = g_j(x), \quad j = 1, 2, \dots, s. \tag{1}$$

Here x, y are points of the manifold Γ; $K_{jh}(x,y)$ are singular kernels, subject to the usual requirements ([1]); T_{jh} are operators completely continuous in the space in which system (1) is studied. We shall assume that the symbolic matrix of this system satisfies the smoothness requirements formulated * in ([1]), and that the symbolic determinant nowhere vanishes; below we shall formulate this last requirement by saying that the symbolic matrix is nonsingular.

In the paper ([2]) (see also ([1])) it is shown that under the indicated conditions the index of system (1) is finite. A. I. Volpert was the first ([3]) to show that the index of a system of two-dimensional singular equations may be different from zero. Important results were obtained by A. I. Volpert in ([4]), where the case when Γ is a two-dimensional sphere is considered. In this paper it is shown that the computation of the index of system (1) can be reduced to the case when its order $s = 2$. Further, for the same case of the two-dimensional sphere, a rule is given for computing the index of system (1): if $s = 2$, then this index is equal to the rotation ** of any one of the rows of the symbolic matrix ***.

In the present note we shall extend A. I. Volpert' s rule to the case $\Gamma = E_2$ and formulate some considerations relating to the more general case.

2°. One may state an assertion generalizing the above-mentioned assertion of A. I. Volpert:

If $m \geq 2$ and $s > m$, then one can construct a system of order m, in which the integration is extended over the same manifold Γ and which has the same index *as system (1).*

We do not give the proof here for lack of space. In accordance with the stated assertion, below, considering the case of a two-dimensional manifold Γ, we shall assume that $s = 2$.

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* We take the opportunity to correct a misprint distorting the sense, which has crept into ([1]), pp. 153 and 158; in Theorems 7.31 and 1.33 it should read: the symbol
$\Phi_A(x, \theta) \mathrel{\hat{\in}} W_2^{(l)}(S)$, where $l \geq \dfrac{m-3}{2}$ for $p \geq 2$ and $l \geq \dfrac{m-1}{p} + 2$ for $p < 2$.

** In the terminology of M. A. Krasnosel' skii ([5]).

*** A. I. Volpert' s results were obtained under rather stringent restrictions on the smoothness of the symbolic matrix. These restrictions are easily weakened by using the results of the book ([1]).

3°. For brevity, let us agree to call the index of a singular system the index of its symbolic matrix. As is known, the problem of the index of a symbolic matrix is sufficiently solved under the assumption that this matrix is unitary— this follows at once from the obvious fact that the index of the adjoint matrix is equal to zero, and from V. F. Atkinson' s theorem ([6]) on the index of a product of operators.

Let Γ be a two-dimensional manifold. A unitary matrix of the second order may be represented in the form

$$\begin{pmatrix} \cos\alpha e^{i\beta} & \sin\alpha e^{i\gamma} \\ -\sin\alpha e^{i(\beta+\delta-\gamma)} & \cos\alpha e^{i\delta} \end{pmatrix}, \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are real; in our case these quantities are functions defined on the set $\tilde{\Gamma}$ of line elements of the manifold Γ. The elements of the matrix (2) must satisfy the smoothness conditions mentioned in item 1°. In that case the function $e^{i(\beta+\delta)}$, equal to the determinant of the matrix (2), satisfies the same smoothness conditions. The matrix (2) may be decomposed into the product

$$\begin{pmatrix} \cos\alpha e^{i\beta} & \sin\alpha e^{i\gamma} \\ -\sin\alpha e^{i(\beta+\delta-\gamma)} & \cos\alpha e^{i\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\beta+\delta)} \end{pmatrix} \begin{pmatrix} \cos\alpha e^{i\beta} & \sin\alpha e^{i\gamma} \\ -\sin\alpha e^{-i\gamma} & \cos\alpha e^{-i\beta} \end{pmatrix}.$$

The index of the first matrix on the right is equal to zero, since the corresponding singular system splits into two separate equations with symbols 1 and $e^{i(\beta+\delta)}$, which do not vanish. The matter is reduced to computing the index of the matrix with determinant equal to one:

$$\begin{pmatrix} \cos\alpha e^{i\beta} & \sin\alpha e^{i\gamma} \\ -\sin\alpha e^{-i\gamma} & \cos\alpha e^{-i\beta} \end{pmatrix}. \tag{3}$$

4°. For the time being assume that the two-dimensional manifold Γ is connected. Consider the four-component vector

$$(\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ \sin\alpha\cos\gamma,\ \sin\alpha\sin\gamma) \tag{4}$$

and denote by l its rotation. Suppose that $l = 0$. Then ([7]) the vector (4) is homotopic to the vector $(1, 0, 0, 0)$; this means that there exist continuous functions (which may be regarded as sufficiently smooth) $\varphi_1(\tau, \lambda)$ and $\varphi_2(\tau, \lambda)$, where τ is a variable line element of the manifold Γ and $\lambda \in [0, 1]$, possessing the following properties: $|\varphi_1(\tau, \lambda)|^2 + |\varphi_2(\tau, \lambda)|^2 > 0$, $\varphi_1(\tau, 0) = 1$, $\varphi_2(\tau, 0) = 0$, $\varphi_1(\tau, 1) = \cos\alpha e^{i\beta}$, $\varphi_2(\tau, 1) = \sin\alpha e^{i\gamma}$. But then the matrix

$$\begin{pmatrix} \varphi_1(\tau, \lambda) & \varphi_2(\tau, \lambda) \\ -\overline{\varphi_2(\tau, \lambda)} & \overline{\varphi_1(\tau, \lambda)} \end{pmatrix}$$

homotopically connects the matrix (3) with the identity matrix in the class of sufficiently smooth nonsingular matrices. Thus, if the rotation of the vector generated by the first row of the matrix (3) (briefly: the rotation of the first row of the matrix (3)) is zero, then this matrix is homotopic to the identity, and its index is zero. Hence (cf. ([4])) it follows that, in the general case,

$$\operatorname{Ind}\Phi = \mu l, \tag{5}$$

where Φ is the symbolic matrix of the system (1), l is the rotation of the first row of the matrix (3), and μ is a constant that depends only on the manifold Γ.

The constant $\mu = 1$ if Γ is a two-dimensional sphere ([4]). It is not hard to see that $\mu = 1$ also in the case when $\Gamma = E_2$*: it suffices to consider any symbo-

* The results of paper ([4]), apparently, do not carry over directly to the case of the Euclidean plane, since there exist symbols that are continuous on the set of line elements of the plane but become discontinuous when the plane is mapped onto the sphere. Such, for example, is the symbol $e^{i\theta}$, where θ is the angle of inclination of the line element to the axis x_1.

logical matrix which, under stereographic projection of the plane onto the sphere, remains continuous; for such (and consequently for any) matrix $\mu = 1$.

5°. Let us derive one auxiliary formula. Consider the topological group G of symbolic matrices defined on $\tilde{\Gamma}$, with the naturally given topology. The dimension m of the manifold Γ will here be considered arbitrary. Let Φ be an arbitrary matrix of the group G. Suppose that there exist N homomorphisms $l_i(\Phi)$, $i = 1, 2, \ldots, N$, of the group G into the group of integers, and suppose that there exist matrices $\Phi_k \in G$, $k = 1, 2, \ldots, N$, such that $l_j(\Phi_k) = \delta_{jk}$. Suppose, finally, that there exists one more homomorphism l_0 of the group G into the group of integers such that if, for some matrix Φ_*, the relations $l_j(\Phi_*) = 0$ $(j = 1, 2, \ldots, N)$ hold, then $l_0(\Phi_*) = 0$. Then

$$l_0(\Phi) = \sum_{j=1}^{N} \mu_j l_j(\Phi), \tag{6}$$

where μ_j are constants. For the proof, put $\mu_j = l_0(\Phi_j)$. Next take an arbitrary matrix $\Phi \in G$ and denote $l_j(\Phi) = q_j$. Then

$$l_j(\Phi\Phi_1^{-q_1}\Phi_2^{-q_2} \ldots \Phi_N^{-q_N}) = q_j - \sum_{k=1}^{N} \delta_{jk} q_k = 0, \qquad j = 1, \ldots, N.$$

Hence $l_0(\Phi\Phi_1^{-q_1}\Phi_2^{-q_2} \ldots \Phi_N^{-q_N}) = 0$, which is equivalent to relation (6).

6°. Let us now consider system (1) under the assumption that Γ is a two-dimensional disconnected manifold. On the group G of symbolic matrices there are defined homomorphisms $l_j(\Phi)$, $j = 1, 2, \ldots, n$, each of which is equal to the rotation of the first row of the corresponding matrix (3) on the surface Γ_j. As Φ_k we take a matrix which, on the surfaces Γ_j, $j \neq k$, coincides with the identity, and on the surface Γ_k has rotation of the first row equal to one. The computation of $\operatorname{Ind}\Phi$ reduces, as above, to the computation of the index of a matrix which on each of the surfaces Γ_j has the form (3). Suppose that $l_j(\Phi_*) = 0$, $j = 1, 2, \ldots, n$. Then on each of the surfaces Γ_j, and consequently also on Γ, the matrix (3) is homotopic to the identity; whence it follows that $\operatorname{Ind}\Phi_* = 0$. Now formula (6) is valid for $\operatorname{Ind}\Phi$. Taking as Φ a matrix equal to the identity on the surfaces Γ_k, $k \neq j$, we find that the coefficient μ_j has the same value as in the case when Γ reduces to a single surface Γ_j. Thus, the index of a singular system is equal to the sum of the indices which this system has on each of the components Γ_j of the manifold Γ. In this form our assertion is true for a manifold of any dimension. In the particular case when all Γ_j are two-dimensional spheres (one of them may degenerate into a plane), $\mu = 1$ and

$$\operatorname{Ind}\Phi = \sum_{j=1}^{n} l_j(\Phi).$$

7°. Let us note several simple cases when the index of a singular system is equal to zero; here there is no need to assume that $s = m$.

A. Let Γ be either a two-dimensional plane or a two-dimensional torus, so that a regular coordinate net can be introduced on Γ. In this case the space $\tilde{\Gamma}$ of linear elements is the direct product of the surface Γ and the unit circle; each linear element is determined by specifying a point $\zeta = e^{i\theta}$ of the unit circle and a point

$x \in \Gamma$; the symbolic matrix of system (1) can be written in the form $\Phi(x, \zeta)$. Fixing x, one may apply to the matrix $\Phi(x, \zeta)$ the left standard factorization ([8])

$$\Phi(x, \zeta) = \Phi_+(x, \zeta)\Phi_0(x, \zeta)\Phi_-(x, \zeta), \tag{7}$$

where $\Phi_+(x, \xi)$ and $\Phi_-(x, \xi)$ admit analytic continuation respectively inside and outside the unit circle of the ξ-plane and have determinants that do not vanish in the indicated domains, while $\Phi_0(x, \xi)$ is a diagonal matrix:

$$\Phi_0(x, \xi) = \left(\xi^{\chi_j}\delta_{jk}\right)_{j,k=1}^{s};$$

the numbers χ_j, called the left partial indices of the matrix $\Phi(x, \xi)$, are integers and do not depend on ξ. The right standard factorization and the right partial indices are introduced analogously ([8]). If the left (or right) partial indices of a matrix do not depend on x, then its index is equal to zero. In particular, the index of a symbolic matrix independent of the pole is equal to zero on the torus.

B. Factorization can sometimes also be applied to the symbolic matrix of a multidimensional system. Suppose that on Γ one can introduce a regular coordinate net. The symbolic matrix has the form $\Phi(x, \theta)$, where $x \in \Gamma$, and $\theta = (\theta_1, \theta_2, \ldots, \theta_{m-1})$ is a point of the unit $(m-1)$-dimensional sphere, with the coordinate θ_{m-1} varying within $[0, 2\pi]$. Setting $\xi = e^{i\theta_{m-1}}$ and fixing the remaining arguments, one can apply the standard factorization to the matrix $\Phi(x, \theta)$. If in this case the left (or right) partial indices are all equal to zero, then $\operatorname{Ind}\Phi = 0$.

C. Let τ be an arbitrary linear element of the manifold Γ (its dimension may be arbitrary), and let $\Phi(\tau)$ be the symbolic matrix of system (1). Put $\Phi(\tau) = \Phi_1(\tau) + i\Phi_2(\tau)$, where the matrices Φ_1 and Φ_2 are self-adjoint. If at least one of these matrices is definite, then $\operatorname{Ind}\Phi = 0$.

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Note: Figure translations are in progress. See original paper for figures.