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Abstract

Full Text

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ON CHAINS OF MATRIX NORMS

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Let $\mathfrak{M}_n$ be the ring of all real square matrices of order n. Consider the set of matrix norms [(1)] in $\mathfrak{M}_n$ with the relation of partial order defined as the fulfillment of the inequality

$$\|A\|_1 \leqslant \|A\|_2 \tag{1}$$

for all $A \in \mathfrak{M}_n$. When condition (1) is fulfilled, one says that the norm $\|A\|_2$ is a majorant for $\|A\|_1$ (and $\|A\|_1$ is a minorant for $\|A\|_2$). If, moreover, $\|A\|_1 < \|A\|_2$ for at least one matrix A, then we shall call $\|A\|_2$ a strict majorant for $\|A\|_1$ (respectively, $\|A\|_1$ a strict minorant for $\|A\|_2$).

Next, consider the set of chains* of matrix norms. This set is partially ordered by inclusion and is inductive. By Zorn' s lemma there exists a maximal chain Z_0. It is easy to see that in fact there even exists a continuum of different maximal chains. The question arises as to the possible structure of maximal chains. In the present note we establish the following result, stated as a conjecture by Yu. I. Lyubich.

Theorem. *All maximal chains are similar** to the half-line* $[0, \infty)$.

We shall precede the proof of this theorem by two lemmas, which are also of independent interest.

We shall call a pseudonorm such a functional in $\mathfrak{M}_n$ which has all the properties of a matrix norm, with the possible exception of the property of positive definiteness. It can easily be proved that

Lemma 1. *Every pseudonorm not identically equal to zero is a matrix norm.*

Next, the following lemma on an "intermediate norm" holds.

Lemma 2. *Let $\|A\|_1$ and $\|A\|_2$ be two matrix norms, and let $\|A\|_2$ be a strict majorant for $\|A\|_1$. Then there exists an intermediate matrix norm $\|A\|$, i.e., a norm which is a strict majorant for $\|A\|_1$ and a strict minorant for $\|A\|_2$.*

Proof. Put $\|A\|' = \frac{1}{2}(\|A\|_1 + \|A\|_2)$. The functional $\|A\|'$ obviously has all the properties of a matrix norm, except, perhaps, the ring property $\|AB\|' \leqslant \|A\|' \cdot \|B\|'$. If this property is fulfilled, then $\|A\|'$ will be the required intermediate norm. In the contrary case there exist two matrices B and C such that $\|BC\|' > \|B\|' \cdot \|C\|'$. Put

$$\|A\|_0 = \max_{U\neq 0} \frac{\|AU\|'}{\|U\|'}.$$

Obviously, $\|A\|_0$ is a matrix norm, and moreover

$$\|B\|_0 \geqslant \frac{\|BC\|'}{\|C\|'} > \|B\|' \geqslant \|B\|_1.$$

This means that the matrix norm $\|A\| = \max(\|A\|_0, \|A\|_1)$ is a strict majorant for $\|A\|_1$. We shall show that this norm is a strict

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* In the sense of the introduced partial order.
** In the usual sense for the theory of ordered sets [(2)].

minorant for $\|A\|_2$. Indeed,

$$\|A\|_0 = \max_{U\neq 0} \frac{\|AU\|}{\|U\|} \leq \max_{U\neq 0} \frac{\|A\|_1 \cdot \|U\|_1 + \|A\|_2 \cdot \|U\|_2}{\|U\|_1 + \|U\|_2} \leq \|A\|_2,$$

and, if $\|A\|_1 < \|A\|_2$ for some matrix A, then $\|A\|_0 < \|A\|_2$ and $\|A\| < \|A\|_2$. The lemma is proved.

Let us proceed to the proof of the theorem. Let Z_0 be some maximal chain. Put $\nu_0 = \inf Z_0$. Obviously, ν_0 is a pseudonorm. Further, since $\nu(E) \geq 1$ for any matrix norm ν, we have $\nu_0(E) \geq 1$. This means that ν_0 is a pseudonorm distinct from the identically zero one and, by Lemma 1, is a matrix norm. Since Z_0 is a maximal chain, $\nu_0 \in Z_0$, i.e. Z_0 is closed from below. Obviously, the chain Z_0 is open from above. To prove the theorem it is now sufficient[2] to establish the continuity of Z_0 and the existence in Z_0 of a countable dense set. Consider a section (P, Q) of the chain Z_0. Put $\nu_1 = \sup P$, $\nu_2 = \inf Q$. By maximality of the chain, $\nu_1, \nu_2 \in Z_0$. Further, $\nu_2 = \nu_1$, since otherwise, by Lemma 2, there would be an intermediate norm not belonging to the chain Z_0, contrary to the maximality of Z_0. This means that Z_0 is continuous.

Let us construct a countable dense set in Z_0. For this purpose, note that if $\nu_1 \in Z_0$, A is some matrix and $\nu_1(A_0) > \lambda > \nu_0(A_0)$, then there exists a norm $\nu_2 \in Z_0$ such that $\nu_2(A_0) = \lambda$. Indeed, define a section in Z_0 by putting $P = \{\nu/\nu(A_0) \leq \lambda\}$, $Q = \{\nu/\nu(A_0) > \lambda\}$. Since $\nu_1 = \sup P = \inf Q$, it follows that $\nu_2(A_0) = \lambda$ and, moreover, $\nu_2 \in Z_0$.

Now let $\mathfrak{M} = \{A_k\}_{k=1}^{\infty}$ be a countable dense set of matrices in $\mathfrak{M}_n$. Put

$$a_k = \nu_0(A_k), \qquad b_k = \sup_{\nu\in Z_0} \nu(A_k)$$

(where it may turn out that $b_k = \infty$). Further, let $R_k = \{r_{kn}\}_{n=1}^{\infty}$ $(k = 1, 2, ...)$ be a countable dense set on the half-interval $[a_k, b_k)$. Denote by ν_{kn} one of those norms $\nu \in Z_0$ for which $\nu(A_k) = r_{kn}$. The set of norms $\{\nu_{kn}\}_{k,n=1}^{\infty}$ is dense in Z_0. Indeed, let $\nu_1, \nu_2 \in Z_0$, and let the norm ν_1 be a strict majorant for ν_2. Then there exists a matrix $A_k \in \mathfrak{M}$ such that $\nu_1(A_k) > \nu_2(A_k)$. Let $\nu_1(A_k) > r_{kn} > \nu_2(A_k)$ $(r_{kn} \in R_k)$. Then the matrix norm ν_{kn} is intermediate for ν_1 and ν_2. Thus, the set of norms $\{\nu_{kn}\}_{k,n=1}^{\infty}$ is a countable dense subset of the chain Z_0. The theorem is proved.

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CITED LITERATURE

1. V. N. Faddeeva, *Computational Methods of Linear Algebra*, Moscow–Leningrad, 1950.

2. F. Hausdorff, *Set Theory*, Moscow–Leningrad, 1937.

Note: Figure translations are in progress. See original paper for figures.