ON THE METHOD OF TANGENT HYPERBOLAS

For the approximate solution of the equation

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Abstract

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MATHEMATICS

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ON THE METHOD OF TANGENT HYPERBOLAS

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1. Let $P(x)$ be a nonlinear operator acting from the ball $D(x_0, R) = \{x : \|x - x_0\| < R\}$ of a Banach space X into a Banach space Y. For the approximate solution of the equation

$$P(x) = 0 \tag{1}$$

various methods are known ($^{1-3}$). However, in many cases the method of tangent hyperbolas, proposed in (^{4}), gives better convergence than other methods.

The successive approximations of the method of tangent hyperbolas are defined from the recurrence relations

$$x_{n+1} = x_n - \theta_n \Gamma_n P(x_n), \qquad n = 0, 1, 2, \dots, \tag{2}$$

where $\theta_n = [I - \frac{1}{2}\Gamma_n P''(x_n)\Gamma_n P(x_n)]^{-1}$, $\Gamma_n = [P'(x_n)]^{-1}$.

In the present paper, by means of the majorant principle, both new convergence conditions for the iterative process (2) and convergence conditions for the process considered by us,

$$x'_{n+1} = x'_n - Q_n \Gamma_0 P(x'_n), \qquad n = 0, 1, 2, \dots, \tag{3}$$

are established, where $Q_n = [I - \frac{1}{2}\Gamma_0 P''(x_0)\Gamma_0 P(x'_n)]^{-1}$, representing a modification of the method of tangent hyperbolas.

2. Consider the equation

$$\varphi(t) = 0, \tag{4}$$

where $\varphi(t)$ is a real function given on the interval $[t_0, t']$, $t' = t_0 + r < t_0 + R$. We shall assume that the function $\varphi(t)$ is three times continuously differentiable on $[t_0, t']$, and that the operator $P(x)$ is three times continuously differentiable with respect to Gâteaux (^{5}) in the closed ball $\overline{D}(x_0, r)$.

We shall say that equation (4) majorizes equation (1) if the following conditions are satisfied:

1°. $\|P(x_0)\| \le \varphi(t_0)$.

2°. There exists the operator $\Gamma_0 = [P'(x_0)]^{-1}$, and moreover $\varphi'(t_0) \ne 0$ and

$$\|\Gamma_0\| \le -\frac{1}{\varphi'(t_0)}.$$

3°. $\|P''(x_0)\| \le \varphi''(t_0)$ and $\|P'''(x)\| \le \varphi'''(t)$, when $\|x - x_0\| \le t - t_0 \le t' - t_0$.

For the real equation (4), the iterative processes (2) and (3) take, respectively, the form

$$t_{n+1} = t_n - \frac{2\varphi'(t_n)\varphi(t_n)}{2[\varphi'(t_n)]^2 - \varphi''(t_n)\varphi(t_n)}, \qquad n = 0, 1, 2, \dots; \tag{5}$$

$$t'_{n+1} = t'_n - \frac{2\varphi'(t_0)\varphi(t'_n)}{2[\varphi'(t_0)]^2 - \varphi''(t_0)\varphi(t'_n)}, \qquad n = 0, 1, 2, \dots. \tag{6}$$

3. In this and in the following sections we shall establish two general theorems on the convergence of the processes (2) and (3). In doing so we shall assume-

that equation (4) has solutions belonging to the interval (t_0, t'), and let us denote the smallest of them by t^*.

Theorem 1. Suppose the following conditions are satisfied:

1) equation (4) majorizes equation (1);
2) $\varphi''(t)\varphi(t)[\varphi'(t)]^{-2} \leqslant \sigma < 2$ for $t \in [t_0, t^*]$.

Then the following assertions hold:

1) there exists a solution x^* of equation (1), belonging to the ball $\widetilde{D}(x_0, r)$, where $r = t^* - t_0$, to which process (2) converges;
2) process (5) converges to the root t^*;
3) the rate of convergence of process (2) is determined by the inequality

$$\|x^* - x_n\| \leqslant t^* - t_n. \tag{7}$$

We give the plan of the proof. First it is established that

$$\|\Gamma_0 P''(x_0)\Gamma_0 P(x_0)\| < 2,$$

and hence there exists an operator θ_0, for which the estimate

$$\|\theta_0\| \leqslant \frac{2[\varphi'(t_0)]^2}{2[\varphi'(t_0)]^2 - \varphi''(t_0)\varphi(t_0)}$$

is derived.

This estimate leads to the inequality

$$\|x_1 - x_0\| \leqslant t_1 - t_0, \tag{8}$$

where $t_1 \leqslant t^*$.

Using now an analogue of Taylor' s formula

$$P(x_1) = P(x_0) + P'(x_0)(x_1 - x_0) + \frac{1}{2}P''(x_0)(x_1 - x_0)^2 +$$

$$+\frac{1}{2}\int_{x_0}^{x_1} P'''(x)(x_1 - x)^2\, dx$$

(on integrals in a Banach space see, for example, (^{6},^{7})), we find that

$$P(x_1) = \frac{1}{4}P''(x_0)\Gamma_0 P''(x_0)\Gamma_0 P(x_0)(x_1 - x_0)^2 + \frac{1}{2}\int_{x_0}^{x_1} P'''(x)(x_1 - x)^2\, dx.$$

Hence, and from other relations, it follows that

$$\|P(x_1)\| \leqslant \varphi(t_1).$$

It is not difficult to see that $\|\Gamma_0[P'(x_0) - P'(x_1)]\| < 1$, whence it follows that the operator $H = \Gamma_0 P'(x_1)$ has an inverse H^{-1}, for which the estimate

$$\|H^{-1}\| \leqslant \frac{\varphi'(t_0)}{\varphi'(t_1)}$$

is valid.

Consequently, there exists the operator $\Gamma_1 = H^{-1}\Gamma_0$, and

$$\|\Gamma_1\| \leqslant -[\varphi'(t_1)]^{-1}.$$

In view of (8), the ball $D(x_1, \tilde{t} - t_1) \subset D(x_0, \tilde{t} - t_0)$, where $\tilde{t} \in [t_1, t^*]$, so that conditions 1°–3° are satisfied when x_0 is replaced by x_1 and t_0 by t_1. By induction it is established that conditions 1°–3° are satisfied when x_0 is replaced by

on x_n and t_0 to t_n for any natural n. In this case it is obtained that $t_n \leq t_{n+1}$, $\varphi(t_n) \geq 0$, and

$$\|P(x_n)\| \leq \varphi(t_n), \qquad n = 0, 1, 2, ... \tag{9}$$

To complete the proof it is established that the sequence $\{t_n\}$ converges to t^* and

$$\|x_{n+p} - x_n\| \leq t_{n+p} - t_n, \tag{10}$$

i.e., that $\{x_n\}$ is a fundamental sequence. Hence, by virtue of (9), the existence of a solution x^* of equation (1) follows, and from (10) the estimate (7) follows.

4. The convergence of the modified process of tangent hyperbolas is established by the following proposition.

Theorem 2. *Suppose that the following conditions are fulfilled:*

1) *equation (4) majorizes equation (1);*
2) $\varphi''(t_0)\varphi(t_0)[\varphi'(t_0)]^{-2} \leq \sigma < 2$.

Then the following assertions hold:

1) *equation (1) has a solution* $x^* \in \widetilde{D}(x_0, r)$, *where* $r = t^* - t_0$, *to which process (3) converges;*
2) *process (6) converges to the root* t^* *of equation (4);*
3) *the rate of convergence of process (3) is determined by the inequality*

$$\|x^* - x'_n\| \leq t^* - t'_n.$$

In proving this theorem, approximately the same plan is used as in the proof of Theorem 1, but the computations differ. Essential changes are introduced in the proof of the inequality

$$\|P(x'_2)\| \leq \varphi(t'_2). \tag{11}$$

By using an analogue of Taylor’ s formula and the equalities

$$P'(x'_1)(x'_2 - x'_1) = P'(x_0)(x'_2 - x'_1) + \int_{x_0}^{x'_1} P''(x)(x'_2 - x'_1)\, dx,$$

$$P''(x_1')(x_2'-x_1')^2 = P''(x_0)(x_2'-x_1')^2 + \int_{x_0}^{x_1'} P'''(x)(x_2'-x_1')^2\,dx,$$

one derives the formula

$$P(x_2') = \frac{1}{4}P''(x_0)\Gamma_0 P''(x_0)\Gamma_0 P(x_1')(x_2'-x_1')^2 +$$

$$+\int_{x_0}^{x_1'} P''(x)(x_2'-x_1')\,dx + \frac{1}{2}\int_{x_0}^{x_1'} P'''(x)(x_2'-x_1')^2\,dx + \frac{1}{2}\int_{x'1}^{x_2'} P'''(x)(x_2'-x)^2\,dx,$$

which leads to inequality (11).

5. With the aid of Theorems 1 and 2 we establish the following propositions, convenient for applications.

Theorem 3. *Suppose that at some point $x_0 \in X$ the following conditions are fulfilled:*

1) $\|P(x_0)\| \le \delta$;
2) *there exists an operator* $\Gamma = [P'(x_0)]^{-1}$, *and* $\|\Gamma\| \le B$;
3) *in the domain* $G = \{x : \|x - x_0\| \le t^*\}$, *where* t^* *is the smallest positive solution of the equation*

$$\varphi(t) \equiv \frac{1}{6}Nt^3 + \frac{1}{2}Mt^2 - B^{-1}t + \delta = 0,$$

the inequalities hold

$$M \geqslant \|P''(x_0)\|, \qquad N \geqslant \sup_{x\in G}\|P'''(x)\|;$$

4) $h = MB^2\delta \leqslant \dfrac{1}{2+\gamma}$, where $\gamma = NB^{-1}M^{-2}$.

Then there exists a solution x^* of equation (1), to which process (2) converges; moreover, the rate of convergence is determined by the inequality
$\|x^* - x_n\| \leqslant t^* - t_n$, where t_n is determined by process (5) with $t_0 = 0$, converging to t^*.

Theorem 4. Suppose that conditions 1), 2), 3) of Theorem 3 and the condition

$$4) \quad h = MB^2\delta \leqslant \frac{1}{3}\gamma^{-2}\big[(1+2\gamma)^{3/2} - (1+3\gamma)\big].$$

are satisfied. Then there exists a solution x^* of equation (1), to which process (3) converges; moreover, the rate of convergence is determined by the inequality $\|x^*-x'_n\| \leqslant t^*-t'_n$, where t'_n is determined by process (6) with $t_0 = 0$, converging to t^*.

6. Let us give the simplest example. For the equation

$$x^5 + x^4 + x^3 + x^2 - 10x + 1 = 0,$$

choosing $x_0 = 0$, we have $\gamma = 8.952$, $h = 0.004$, $\dfrac{1}{2+\gamma} > 0.091$. Hence, by Theorems 3 and 4, the implementability of processes (2) and (3) follows.

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