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## Abstract

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# APPLICATION OF THE METHOD OF GRIDS TO A HYPERBOLIC SYSTEM OF QUASILINEAR EQUATIONS IN THE PLANE

*(Presented by Academician S. L. Sobolev on 13 VIII 1962)*

In the strip $(0 \le t \le t^0,\ 0 \le x \le x^0,\ -\infty < u_1, \dots, u_n < \infty)$ let us consider the system of equations

$$\frac{\partial u_1(t,x)}{\partial t} = \lambda_1(t,x,u)\frac{\partial u_1(t,x)}{\partial x} + f_1(t,x,u),$$
$$\frac{\partial u_2(t,x)}{\partial t} = \lambda_2(t,x,u)\frac{\partial u_2(t,x)}{\partial x} + f_2(t,x,u), \tag{1}$$

where $u_1, u_2, u = \left\|\begin{matrix} u_1 \\ u_2 \end{matrix}\right\|$, $f_1, f_2$ are column matrices; $\lambda_1, \lambda_2$ are diagonal matrices, with the diagonal elements of $\lambda_1$ nonnegative, and positive for $x = x^0$, while the diagonal elements of $\lambda_2$ are nonpositive, and negative for $x = 0$, with initial conditions

$$u(0,x) = \varphi(x) \tag{2}$$

and boundary conditions

$$u_1(t,x^0) = \alpha_1(t, u_2(t,x^0)) + \int_0^t \beta_1(t,\tau,u(\tau,x^0))\,d\tau,$$
$$u_2(t,0) = \alpha_2(t, u_1(t,0)) + \int_0^t \beta_2(t,\tau,u(\tau,0))\,d\tau. \tag{3}$$

Analogous systems of linear equations were studied in papers ($^{1-3}$). In studying classical and generalized solutions we shall need compatibility conditions:

$$\varphi_1(x^0) = \alpha_1(0, \varphi_2(x^0)),$$

$$\varphi_2(0) = \alpha_2(0, \varphi_1(0)); \tag{4}$$

$$\lambda_1(0, x^0, \varphi(x^0))\varphi_1'(x^0) + f_1(0, x^0, \varphi(x^0)) = \frac{\partial \alpha_1}{\partial t}\bigg|_{t=0,\ u_2=\varphi_2(x^0)} + \beta_1(0, 0, \varphi(x^0)), \tag{5}$$

$$\lambda_2(0, 0, \varphi(0))\varphi_2'(0) + f_2(0, 0, \varphi(0)) = \frac{\partial \alpha_2}{\partial t}\bigg|_{t=0,\ u_1=\varphi_1(0)} + \beta_2(0, 0, \varphi(0)).$$

To replace the corresponding boundary-value problem by a difference one, we construct a grid with step $h$ in $x$ and $k$ in $t$. Then the difference system is written as follows:

$$\begin{gathered} u_1^{i+1\,j} = \varkappa\lambda_1^{ij} u_1^{i\,j+1} + (\varepsilon_1 - \varkappa\lambda_1^{ij})u_1^{ij} + kf_1^{ij}, \\ u_2^{i+1\,j} = -\varkappa\lambda_2^{ij} u_2^{i\,j+1} + (\varepsilon_2 + \varkappa\lambda_2^{ij})u_2^{ij} + kf_2^{ij}; \\ u_1^{i+1\,\nu} = \alpha_1^{i+1} + k\sum_{p=0}^{i} \beta_1^{i+1\,p}, \qquad u_2^{i+1\,0} = \alpha_2^{i+1} + k\sum_{p=0}^{i} \beta_2^{i+1\,p}, \end{gathered} \tag{6}$$

where $\varkappa = kh^{-1}$; $\varepsilon_1, \varepsilon_2$ are unit matrices.

For a matrix $A = \|a_{pq}\|$ the norm $|A|$ is equal to $\max_q \sum_{p=1}^{p_0} |a_{pq}|$. The matrix $A(t, x, u)$ satisfies the Lipschitz condition ($A \in \mathrm{Lip}$), if for every $a > 0$ there is a $K$ such that

$$|A(t, x, u) - A(\bar{t}, \bar{x}, \bar{u})| \le K(|t - \bar{t}| + |x - \bar{x}| + |u - \bar{u}|)$$

for $|u|, |\bar{u}| \le a$. If in the rectangle $\Pi^* = \Pi_{t^*}(0 \le t \le t^*,\ 0 \le x \le x^0)$ the integral relation

$$\iint_{\Pi^*} \left[\left(\frac{\partial v}{\partial t} - \frac{\partial v\lambda}{\partial x}\right)u + vf\right] d\Pi = \int_0^{x^0} vu\Big|_0^{t^*} dx - \int_0^{t^*} v\lambda u\Big|_0^{x^0} dt,$$

holds for $u, \lambda, f \in \mathrm{Lip}$, where $v$ is any continuously differentiable row matrix, then we shall say that $u$ is a generalized solution of system (1).

**Theorem 1.** If $\lambda, f, \varphi, \alpha, \beta$ have first partial derivatives satisfying the Lipschitz condition, and the compatibility conditions (4)–(5) are fulfilled, then there exists a $t^* > 0$ such that in $\Pi^*$ there exists a classical solution of the boundary-value problem (1)–(3), and moreover $du/dt$, $du/dx \in \mathrm{Lip}$.

**Theorem 2.** If $\lambda, f, \varphi, \alpha, \beta \in \mathrm{Lip}$ and the compatibility conditions (4) are fulfilled, then there exists a $t^* > 0$ such that in $\Pi^*$ there exists a generalized solution of the boundary-value problem (1)–(3), and moreover $u \in \mathrm{Lip}$.

The proofs of these theorems are based on the following lemmas.

**Lemma 1.** If $\lambda, f, \varphi, \alpha, \beta \in \mathrm{Lip}$ and the compatibility conditions (4) are fulfilled, then there exist $U_1, U_2$ and $t^2 > 0$ such that, if for the mesh the relations $\varkappa|\lambda^{ij}| \leq 1,\ 0 \leq a \leq \varkappa|\lambda_1^i|^{-1},\ a \leq \varkappa|\lambda_2^i|$ are fulfilled in $\Pi_{t^1}$, then $|u^{ij}| \leq U_1,\ |\Delta_1 u^{ij}| \leq U_2,\ |\Delta_2 u^{ij}| \leq U_2$ in $\Pi_{t^*}$, where $t^* = \min\{t^1, t^2\}$, $\Delta_1 u^{ij} = (u^{i+1,j} - u^{ij})k^{-1}$, $\Delta_2 u^{ij} = (u^{i,j+1} - u^{ij})h^{-1}$.

**Lemma 2.** If $\lambda, f, \varphi, \alpha, \beta$ have first partial derivatives satisfying the Lipschitz condition, and the compatibility conditions (4)–(5) are fulfilled, then there exist $U_3$ and $t^3 > 0$ such that, if for the mesh the relations $\varkappa|\lambda^{ij}| \leq 1,\ 0 < a \leq \varkappa|\lambda_1^i|^{-1},\ a \leq \varkappa|\lambda_2^i|$ are fulfilled in $\Pi_{t^2}$, then $|\Delta_1^2 u^{ij}| \leq U_3,\ |\Delta_1 \Delta_2 u^{ij}| \leq U_3,\ |\Delta_2^2 u^{ij}| \leq U_3$ in $\Pi_{t^*}$, where $t^* = \max\{t^2, t^3\}$.

**Lemma 3.** If $\lambda, f \in \mathrm{Lip}$ and a sequence $u^p$ of solutions of system (6) for meshes whose step tends to zero converges uniformly in $\Pi^*$ to $u$, and $|\Delta_1 u^p| \leq U_2,\ |\Delta_2 u^p| \leq U_2$, then $u$ is a generalized solution of system (1).

**Theorem 3.** If $\lambda, f, \varphi, \alpha, \beta \in \mathrm{Lip}$ and the compatibility conditions (4) are fulfilled, then the generalized solution of the boundary-value problem (1)–(3) is unique.

**Theorem 4.** If $\lambda, \lambda_\varepsilon, f, f_\varepsilon, \varphi, \varphi_\varepsilon, \alpha, \alpha_\varepsilon, \beta, \beta_\varepsilon \in \mathrm{Lip}$, the compatibility conditions (4) are fulfilled, the generalized solution exists in $\Pi^*$, $|\lambda - \lambda_\varepsilon| = O(\varepsilon), \ldots, |\beta - \beta_\varepsilon| = O(\varepsilon)$, then, for sufficiently small $\varepsilon$, from the existence of the generalized solution $u_\varepsilon$ in $\Pi^*$ it follows that $|u - u_\varepsilon| = O(\varepsilon)$.

**Theorem 5.** Let $u$ be a classical solution in $\Pi^*$ of the boundary-value problem (1)–(3), $du/dt,\ du/dx \in \mathrm{Lip}$. If $\lambda, f, \varphi, \alpha, \beta \in \mathrm{Lip}$, then for sufficiently small $h$ there is a $C$ such that from $\varkappa|\lambda^{ij}| \leq 1$ it follows that

$$|u(t_i, x_j) - u^{ij}| \leq Ch.$$

**Theorem 6.** Let $u$ be a generalized solution in $\Pi^*$ of the boundary-value problem (1)–(3). If $\lambda, f, \varphi, \alpha, \beta \in \mathrm{Lip}$, then for sufficiently small $h$ there is a $C$ such that from $\varkappa|\lambda^{ij}| \leq 1$ it follows that

$$|u(t_i, x_j) - u^{ij}| \leq Ch^{1/2}.$$

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*Note: Figure translations are in progress. See original paper for figures.*