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Abstract

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ON THE DEVIATION OF FUNCTIONS BIHARMONIC IN A DISK FROM THEIR BOUNDARY VALUES

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The basic biharmonic problem, to which, in particular, the determination of the stress function in the plane theory of elasticity is reduced, as is known ([1]), consists in determining a function $U(x, y)$ having in a given domain G continuous partial derivatives up to and including the fourth order, which satisfy inside G the equation

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0, \qquad (1)$$

and on the boundary Γ of the domain G the conditions

$$U\Big|_\Gamma = f(s), \qquad \frac{\partial U}{\partial n}\Big|_\Gamma = h(s), \qquad (2)$$

where $f(s)$ and $h(s)$ are prescribed summable functions of the arc s of the contour Γ.

Restricting ourselves to the case when the domain G is a disk, and assuming that $h(s) = 0$, we shall consider here the question of the magnitude of the deviation of the values of the function $U(x, y)$ inside the disk from the corresponding values on the boundary, under various differential-difference properties of $U(x, y)$ on Γ. Various results of this kind for harmonic functions were obtained in papers ([2]-[6]).

Let G be the unit disk, $r = \sqrt{x^2 + y^2} < 1$, $x = r\cos\varphi$, $y = r\sin\varphi$, and $f(r, \varphi) = U(x, y)$.

Theorem 1. For any function $f(r, \varphi)$ biharmonic in the unit disk and satisfying on the boundary the condition

$$\frac{\partial f(r, \varphi)}{\partial r}\Big|_{r=1} = 0,$$

for all $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant r \leqslant 1$, the inequality

$$|f(r, \varphi) - f(1, \varphi)| \leqslant C_1 \omega_2(1 - r), \qquad (3)$$

holds, where C_1 is some absolute constant,

$$\omega_2(t) = \sup_{0<h<t} |f(1,\varphi+h) - 2f(1,\varphi) + f(1,\varphi-h)| \tag{4}$$

is the modulus of smoothness of the function $f(1,\varphi)$.

Theorem 2. Let $f(r,\varphi)$ be a function biharmonic in the disk $r < 1$ and continuous in the closed disk $r \leqslant 1$, satisfying on the boundary the condition

$$\left.\frac{\partial f(r,\varphi)}{\partial r}\right|_{r=1} = 0,$$

and let $\omega(t)$ be the modulus of continuity of the function $f(1,\varphi)$. In that case the inequality

$$\left|\frac{\partial f(r,\varphi)}{\partial \varphi}\right| \leqslant C_2 r \frac{\omega(1-r)}{1-r}, \tag{5}$$

holds, where $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant r < 1$, and C_2 is an absolute constant.

From Theorem 1 there follow:

Corollary 1. If the function $f(1,\varphi)$ is quasi-smooth, i.e., satisfies uniformly in all φ the condition

$$|f(1,\varphi+h) - 2f(1,\varphi) + f(1,\varphi-h)| \leqslant M_1|h|, \tag{6}$$

then for all $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant r < 1$ the inequality

$$|f(r,\varphi) - f(1,\varphi)| \leqslant C_3(1-r). \tag{7}$$

Corollary 2. If the function $f(1,\varphi)$ has a first derivative $f'(1,\varphi)$ satisfying a Lipschitz condition, i.e., the condition

$$|f'(1,\varphi+h) - f'(1,\varphi)| \leqslant \varphi M_2|h|, \tag{8}$$

then for all $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant r < 1$ the inequality

$$|f(r,\varphi) - f(1,\varphi)| \leqslant C_4(1-r)^2. \tag{9}$$

The following two theorems refine estimates (7) and (9). Consider the class of 2π-periodic quasismooth functions $f(1,\varphi)$ (denote it by MH_2) and the class $MW^{(p)}$ ($p > 1$ an integer) of functions $f(1,\varphi)$ of period 2π, having an absolutely continuous derivative of order $(p-1)$ and almost everywhere a derivative of order p, $|f^{(p)}(1,\varphi)| \leqslant M$. Theorems 3 and 4 given below establish, for each $0 \leqslant r < 1$, the exact value of the upper bound

$$\max_{0\leqslant\varphi\leqslant 2\pi}|f(r,\varphi)-f(1,\varphi)| = \Delta(f;r)$$

on these classes.

Theorem 3. For every $0<r<1$ the exact equality holds

$$\sup_{f\in MH_2}\Delta(f;r) = \frac{2M}{\pi}(1-r)+\frac{M}{\pi}\varepsilon_r, \tag{10}$$

where

$$\begin{aligned}\varepsilon_r &= (1-r)^2\ln(1+r)+2\left[\ln\frac{1}{r}-(1-r)\right]\ln(1+r)\\ &\quad+(1-r)^2\ln\frac{1}{1-r}-2\int_r^1\frac{\ln t}{1+t}\,dt-2\int_0^{1-r}\frac{t\ln t}{1-t}\,dt.\end{aligned} \tag{11}$$

Remark 1. From expression (11) it is clear that as $r\to 1$

$$\varepsilon_r = o(1-r).$$

Remark 2. Comparing estimate (10) with the known estimate

$$\sup_{f\in MH_2}\Delta(f;r) = \frac{2M}{\pi}(1-r)\ln\frac{1}{1-r}+M\varepsilon'_r, \quad \text{where } \varepsilon'_r = O(1-r), \tag{12}$$

for harmonic functions, obtained by A. F. Timan in paper ([3]), we see that as $r\to 1$ the right-hand side of (10) is of lower order than (12), despite the fact that the class of biharmonic functions is wider than the class of harmonic functions. This circumstance is explained by the fact that here only those biharmonic functions are considered whose normal derivative on the boundary is equal to zero, and this class, apart from constants, contains no other harmonic functions.

In the following theorem the known Bernoulli numbers appear:

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \quad n\geqslant 0,$$

$$\widetilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}}, \quad n\geqslant 1.$$

Theorem 4. For any integer $p>1$, for each $0<r<1$ the following exact equality holds

$$
\begin{aligned}
\sup_{f \in MW(p)} \Delta(f; r) = M \sum_{i=1}^{p/2} \frac{1}{(2i-1)!} \widetilde{K}_{p-2i+1} \ln^{2i-1} \frac{1}{r} \\
- M \sum_{i=1}^{(p-2)/2} \frac{1}{(2i)!} \widetilde{K}_{p-2i} \ln^{2i} \frac{1}{r} \\
+ \frac{1-r^2}{2} \left[M \sum_{i=1}^{(p-2)/2} \frac{1}{(2i-1)!} K_{p-2i} \ln^{2i-1} \frac{1}{r} - M \sum_{i=0}^{(p-2)/2} \frac{1}{(2i)!} \widetilde{K}_{p-2i-1} \ln^{2i} \frac{1}{r} \right] \\
- M \alpha_r^{(p)} + M \frac{1-r^2}{2} \alpha_r^{(p-1)},
\end{aligned}
\tag{13}
$$

where

$$
\alpha_r^{(p)} = \frac{4}{\pi} \int_r^1 \int_{t_p}^1 \int_{t_{p-1}}^1 \cdots \int_{t_2}^1 \frac{1}{t_1 t_2 \dots t_p} \arctan t_1 \, dt_1 \, dt_2 \dots dt_p,
$$

if p is even, and

$$
\begin{aligned}
\sup_{f \in MW(p)} \Delta(f; r) = M \sum_{i=1}^{(p-1)/2} \frac{1}{(2i-1)!} \widetilde{K}_{p-2i+1} \ln^{2i-1} \frac{1}{r} \\
- M \sum_{i=1}^{(p-1)/2} \frac{1}{(2i)!} K_{p-2i} \ln^{2i} \frac{1}{r} \\
+ \frac{1-r^2}{2} \left[M \sum_{i=1}^{(p-1)/2} \frac{1}{(2i-1)!} K_{p-2i} \ln^{2i-1} \frac{1}{r} - M \sum_{i=0}^{(p-3)/2} \frac{1}{(2i)!} \widetilde{K}_{p-2i-1} \ln^{2i} \frac{1}{r} \right] \\
+ M \beta_r^{(p)} - M \frac{1-r^2}{2} \beta_r^{(p-1)},
\end{aligned}
\tag{14}
$$

where

$$
\beta_r^{(p)} = \frac{2}{\pi} \int_r^1 \int_{t_p}^1 \int_{t_{p-1}}^1 \cdots \int_{t_2}^1 \frac{1}{t_1 t_2 \dots t_p} \ln \frac{1+t_1}{1-t_1} \, dt_1 dt_2 \dots dt_p,
$$

if p is odd;

$$
\sup_{f \in MW(n)} \Delta(f; 0) = M K_p, \qquad p \geqslant 1. \tag{15}
$$

Remark 3. For judging the behavior of the upper bounds (13) and (14), one must keep in mind that as $r \to 1$

$$\ln \frac{1}{r} \simeq 1 - r, \qquad \alpha_r^{(p)} = O\{(1-r)^p\}, \qquad \beta_r^{(p)} = O\left\{(1-r)^p \ln \frac{1}{1-r}\right\},$$

which can be verified directly.

Remark 4. Attention is drawn to the fact that the principal term of the upper bounds (13) and (14) for all integers $p \geqslant 2$ has order $(1-r)^2$. It turns out that such a deviation of $f(r, \varphi)$ from its boundary values is, in order, the best possible for the functions considered here. It can be shown that if a biharmonic function, for

for which

$$\left.\frac{\partial f(r,\varphi)}{\partial r}\right|_{r=1} = 0,$$

is such that, as $r \to 1$,

$$\max_{0 \leqslant \varphi \leqslant 2\pi} |f(r,\varphi) - f(1,\varphi)| = O\{(1-r)^2\},$$

then $f(r, \varphi) \equiv \text{const}$.

In connection with Remark 4, the question arises as to what the function $f(1, \varphi)$ must be in order that the relation

$$\max_{0 \leqslant \varphi \leqslant 2\pi} |f(r,\varphi)| = O\{(1-r)^2\}. \tag{16}$$

hold.

The answer to this question is given by the following theorem:

Theorem 5. *In order that relation (16) hold, it is necessary and sufficient that the function $f(1, \varphi)$ have a first derivative $f'(1, \varphi) \in \mathrm{Lip}\,1$.*

I take this opportunity to express my sincere gratitude to Prof. A. F. Timan for posing the problems and for his constant attention.

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Note: Figure translations are in progress. See original paper for figures.