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Abstract

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SOME CRITERIA FOR ABSOLUTE CESÀRO SUMMABILITY OF MULTIPLE FOURIER SERIES

(Presented by Academician S. N. Bernstein, 16 V 1963)

Thanks to the well-known results of S. N. Bernstein ([1]) and to their subsequent development and generalization in the works of Zygmund, Sas, Salem, Stechkin, Konyushkov, and others, at present there are, in a certain sense, definitive criteria for the absolute convergence of Fourier series of a function of one variable.

A number of works has been devoted to clarifying the conditions that must be satisfied by a function $f(x_1, \dots, x_k)$, periodic in each of its variables, which ensure the absolute convergence or summability of its Fourier series (see, for example, ([2]-[4])). The criteria indicated in these works contain restrictions on the function $f(x_1, \dots, x_k)$ with respect to the totality of all groups of its variables. For the first time, in the work ([6]) of M. F. Timan, criteria are given for the absolute convergence of multiple Fourier series that contain restrictions on the function $f(x_1, \dots, x_k)$ only with respect to each of its variables separately and, owing to this, are less cumbersome and more effective for verification.

Below we present some results concerning absolute Cesàro summability (or $|C; \beta_1, \dots, \beta_k|$-summability) of multiple Fourier series. Moreover, as in the work ([6]), conditions are indicated that contain restrictions on the function $f(x_1, \dots, x_k)$ only with respect to each of the variables separately and ensure the absolute summability of its Fourier series.

Let the series

$$\sum_{n_1, \dots, n_k = 0}^{\infty} \lambda_{n_1, \dots, n_k} A_{n_1, \dots, n_k}(x_1, \dots, x_k) \qquad (1)$$

be the Fourier series of a function $f(x_1, \dots, x_k)$, integrable in the Lebesgue sense and periodic with period 2π in each of its variables, where

$$A_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{i_1=1}^{2} \cdots \sum_{i_k=1}^{2} a_{n_1, \dots, n_k}^{(i_1, \dots, i_k)} \prod_{\nu=1}^{k} \gamma_{i_\nu}(n_\nu x_\nu);$$

$$\gamma_i(nx) = \begin{cases} \cos nx, & \text{for } i = 1, \\ \sin nx, & \text{for } i = 2; \end{cases}$$

$\lambda_{n_1,\dots,n_k} = \dfrac{1}{2^m}$ (m is the number of indices n_ν ($\nu = 1, 2, \dots, k$) equal to zero);

$a^{(i_1,\dots,i_k)}_{n_1,\dots,n_k}$ are the Fourier coefficients of the function $f(x_1, \dots, x_k)$.

Theorem 1*. *If the function $f(x_1, \dots, x_k)$, periodic with period 2π in each of the variables, is such that*

$$\|f(x_1, \dots, x_k)\|_{L_2} = \left\{ \int_0^{2\pi} \cdots \int_0^{2\pi} |f(x_1, \dots, x_k)|^2 \, dx_1 \cdots dx_k \right\}^{1/2} < \infty, \qquad (2)$$

* For a function of one variable (i.e. $k = 1$), Theorem 1 was proved by M. F. Timan, who kindly informed me of this.

and the conditions are satisfied

$$\sum_{n=1}^{\infty} n^{k/2-\beta_\nu-1} \omega_k^{(\nu)} \left(f; \frac{1}{n}\right)_{L_2} < \infty \qquad \left(-1 < \beta_\nu < \frac{1}{2},\ \nu = 1, 2, \dots, k\right), \qquad (3)$$

where

$$\omega_k^{(\nu)}(f; h)_{L_2} = \sup_{|t| \le h} \left\| \sum_{\mu=0}^{k} (-1)^{k-\mu} \binom{k}{\mu} f(x_1, \dots, x_{\nu-1}, x_\nu + \mu t, x_{\nu+1}, \dots, x_k) \right\|_{L_2},$$

then the series (1) is almost everywhere absolutely summable by the method $|C; \beta_1, \dots, \beta_k|$.*
For $\beta_1 = \beta_2 = \dots = \beta_k = 0$, using the Luzin-Danjoy theorem (see (9), Ch. 6), as a special case we obtain Theorem 5 from (9).

Theorem 2. If, for a function $f(x_1, \dots, x_k)$ periodic with period 2π in each of the variables, the conditions

$$\sum_{n=1}^{\infty} \sqrt{\ln n}\, n^{\frac{k-3}{2}} \omega_k^{(\nu)} \left(f; \frac{1}{n}\right)_{L_2} < \infty \qquad (\nu = 1, 2, \dots, k), \qquad (4)$$

are satisfied, then the Fourier series of the function $f(x_1, \dots, x_k)$ is almost everywhere absolutely summable by the method $|C; \frac{1}{2}, \dots, \frac{1}{2}|$.

Theorem 3. If, for a function $f(x_1, \dots, x_k)$ periodic with period 2π in each of the variables, the conditions

$$\sum_{n=1}^{\infty} n^{k-2} \omega_k^{(\nu)} \left(f; \frac{1}{n}\right)_{L_2} < \infty \qquad (\nu = 1, 2, \dots, k), \tag{5}$$

are satisfied, then the Fourier series of the function $f(x_1, \dots, x_k)$ is almost everywhere absolutely summable by the method $|C; \beta_1, \dots, \beta_k|$, where $\beta_\nu > \frac{1}{2}$.

Theorems 1, 2, and 3 give conditions for absolute summability in structural terms of the partial moduli of smoothness of the function.

The following proposition establishes conditions for absolute Cesàro summability in constructive terms.

Theorem 4. If $f(x_1, \dots, x_k) \in L_2$, i.e. satisfies relation (2), then the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{k/2-\beta_\nu-1} E_{n_\nu}(f)_{L_2} < \infty \tag{6}$$

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* A multiple series

$$\sum_{n_1, \dots, n_k}^{\infty} u_{n_1, \dots, n_k}$$

is called $|C; \beta_1, \dots, \beta_k|$-summable ($\beta_i > -1, \ i = 1, 2, \dots, k$), if for every set of indices $n_{\nu_1}, \dots, n_{\nu_m}$ $(m \leq k)$ the conditions

$$\sum_{n_{\nu_1}=1}^{\infty} \cdots \sum_{n_{\nu_m}=1}^{\infty} \frac{\left|\tau_{n_{\nu_1}, \dots, n_{\nu_m}}^{\beta_{\nu_1}, \dots, \beta_{\nu_m}}\right|}{n_{\nu_1} \cdots n_{\nu_m}} < \infty,$$

are fulfilled, where

$$\tau_{n_{\nu_1}, \dots, n_{\nu_m}}^{(\beta_{\nu_1}, \dots, \beta_{\nu_m})} =$$

$$= \frac{\sum_{\mu_{\nu_1}=1}^{n_{\nu_1}} \cdots \sum_{\mu_{\nu_m}=1}^{n_{\nu_m}} A_{n_{\nu_1}-\mu_{\nu_1}}^{\beta_{\nu_1}-1} \cdots A_{n_{\nu_m}-\mu_{\nu_m}}^{\beta_{\nu_m}-1} \mu_{\nu_1} \cdots \mu_{\nu_m} u_{0,\dots,0,\mu_{\nu_1},0,\dots,0,\mu_{\nu_m},\dots,0}}{A_{n_{\nu_1}}^{\beta_{\nu_1}} \cdots A_{n_{\nu_m}}^{\beta_{\nu_m}}},$$

$$A_n^{\beta} = \frac{(\beta+1)\cdots(\beta+n)}{n!}$$

(see, for example, (5)).

ensure almost everywhere absolute $|C;\beta_1,\dots,\beta_k|$-summability of series (1) $(-1 < \beta_\nu < 1/2,\ \nu = 1,2,\dots,k)$; the conditions

$$\sum_{n_\nu=1}^{\infty} \sqrt{\ln n_\nu}\, n_\nu^{\frac{k-3}{2}} E_{n_\nu,\infty}(f)_{L_2} < \infty \tag{7}$$

$$(\nu = 1,2,\dots,k)$$

are sufficient for almost everywhere $|C;1/2,\dots,1/2|$-summability of series (1) and, finally, the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{k-2} E_{n_\nu,\infty}(f)_{L_2} < \infty \tag{8}$$

ensure almost everywhere $|C;\beta_1,\dots,\beta_k|$-summability for $\beta_\nu > 1/2$, $\nu = 1,2,\dots,k$, where $E_{n_\nu,\infty}(f)_{L_2}$ are partial best approximations of the function $f(x_1,\dots,x_k)$—with respect to the variables x_ν respectively $(\nu = 1,2,\dots,k)$ (see ([8])).

We note that conditions (3), (4), (5) are respectively equivalent to conditions (6), (7), (8).

We give one more proposition concerning the question under consideration.

Theorem 5. If $f(x_1,\dots,x_k) \in L_2$ and, for some system of numbers $\alpha_\nu > 0$ $(\nu = 1,2,\dots,k)$ such that $\alpha_1 + \cdots + \alpha_k = 1$, the series

$$\sum_{n_\nu=1}^{\infty} n_\nu^{-\beta_\nu} E_{n_\nu,\infty}^{\alpha_\nu}(f)_{L_2} \qquad (-1/2 < \beta_\nu \le 1/2),$$

converge, then the Fourier series of the function $f(x_1,\dots,x_k)$ is absolutely summable by the method $|C;\beta_1 - 1/2,\dots,\beta_k - 1/2|$.

For $\beta_1 = \beta_2 = \cdots = \beta_k = 1/2$ we obtain, as a special case, Theorem 1 from ([6]).

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Note: Figure translations are in progress. See original paper for figures.