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Abstract

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ON AN INTEGRAL REPRESENTATION OF A VECTOR HOLOMORPHIC IN A BALL

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In this note the simplest boundary-value problems are considered for a vector holomorphic in a ball, and an integral representation is derived for such a vector.

A four-component vector V is called holomorphic if its components, in some domain of three-dimensional Euclidean space E_3, satisfy the elliptic system

$$DV = 0, \tag{1}$$

where $D = \sum_{i=1}^{3} \gamma_i \dfrac{\partial}{\partial x_i}$, and the matrices γ_i have the form (see (^{1,2})):

$$\gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the ball G of unit radius, bounded by the sphere S. Denote by C the equator of the sphere.

In the plane case an analytic function can always be represented (up to an imaginary constant) by an integral of Cauchy type with real density. We obtain an analogous integral representation for a vector holomorphic in a ball. First consider the Dirichlet problem:

I. Find a vector $V(p, u, v, w)$, holomorphic in the ball, Hölder-continuous in $G + S$, and satisfying on S the boundary condition

$$p\big|_S = f_1(x), \qquad w\big|_S = f_4(x), \tag{2}$$

where $f_1(x)$ and $f_4(x)$ are real functions given on S and Hölder-continuous, $f_1(x) \in W_2^{(2)}(S)$, $f_4(x) \in W_2^{(2)}(S)$ (^{8}).

Putting $u + iv = U,\ w + ip = W$, we write system (1) in complex form

$$U_\zeta + W_z = 0,$$

$$U_z - W_{\bar\zeta} = 0, \tag{3}$$

where $\zeta = x_1 + ix_2$; $\dfrac{\partial}{\partial\zeta} = \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}$; $\dfrac{\partial}{\partial\bar\zeta} = \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2}$; $z \equiv x_3$.

Let (r, θ, φ) be the spherical coordinates of the point $x(x_1, x_2, x_3)$. For system (3) a complete system of solutions is constructed:

$$U_{n,m} = r^n Y_n^{m+1}, \qquad n = 0, 1, 2, \dots,$$

$$W_{n,m} = (n + m + 1) r^n Y_n^m, \qquad -n - 1 \le m \le n,$$

where $Y_n^m = P_n^m(\cos\theta)e^{im\varphi}$ are spherical functions of order n (see (^{5})). Now on S: $W|_S = f$; $f(x) = f_4(x) + if_1(x)$. The function $f(x)$ can be expanded in a uniformly convergent series in spherical functions (see (^{6}), p. 380 and (^{5}), p. 319):

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n,m} Y_n^m.$$

Representing W by the series

$$\sum_{n=0}^{\infty} \sum_{m=-n-1}^{n} a_{n,m} W_{n,m}$$

with complex coefficients and comparing coefficients on S, we obtain

$$(n + m + 1) a_{n,m} = f_{n,m}, \qquad -n \leqslant m \leqslant n.$$

Since all the coefficients $a_{n,m}$, except $a_{n,-n-1}$, are determined, $W(x)$ in the ball G is determined uniquely, while $U(x)$ is determined up to the term

$$\sum_{n=0}^{\infty} a_{n,-n-1} Y_n^{-n} = \sum_{n=0}^{\infty} b_n \bar\zeta^n,$$

i.e., in the ball G

$$U(x) = U_0(x) + \overline{\Phi(\zeta)},$$

where $U_0(x)$ is uniquely determined by the function f, and $\Phi(\zeta)$ is an arbitrary analytic function of ζ.

Therefore problem I in the class of functions Hölder-continuous in the closed ball is always solvable, and the solution is determined up to a function conjugate to an analytic one. In particular, the homogeneous problem $\overset{\circ}{\mathrm{I}}$ has an infinite number of linearly independent solutions

$$U(x) = \overline{\Phi(\zeta)}, \qquad W(x) \equiv 0$$

($\Phi(\zeta)$ is any analytic function of ζ).

Of course, problem (1)–(2) may be regarded as ill-posed (see (^{7})); however, Dirichlet problem I will be posed correctly if, to the conditions (2) on S, one adds conditions on C that uniquely determine the analytic function $\Phi(\zeta)$.

It is easy to see how the result changes if to (2) one adds conditions on C of the form $\mathrm{Re}[\zeta^n U]|_C = \varphi(\zeta)$ (see (^{4})), where $\varphi(\zeta)$ is a real Hölder-continuous function. The following problem may be regarded as conjugate to problem I:

I′. Find a vector $V'(p', u', v', w')$ satisfying in G the system

$$D'V' = 0$$

(the prime on the operator D denotes transposition) and on S the boundary condition

$$x_1 p' + x_3 v' - x_2 w' = f_2'(x),$$

$$x_2 p' - x_3 u' + x_1 w' = f_3'(x),$$

$$f_2'(x) + i f_3'(x) = f'(x) \in W_2^{(2)}(S).$$

Analogously to the preceding, we obtain:

In the class of functions representable in $G + S$ by absolutely and uniformly convergent series in ball functions, the homogeneous problem $\overset{\circ}{\mathrm{I}}{}'$ has only the trivial solution, and the nonhomogeneous problem I′ is solvable if the condition

$$\iint_S \Phi(\zeta) f'(x)\, dS_x \tag{4}$$

is satisfied for any analytic function $\Phi(\zeta)$.

Condition (4), written in real form, shows that $f'(x)$ on the sphere must be orthogonal to any solution of the homogeneous problem $\overset{0}{\mathrm{I}}$, i.e., normal solvability of the problem holds.

Normal solvability is preserved (under some additional assumptions) also in the case of a problem with general boundary conditions of the form:

$$\lambda_{i1}p + \lambda_{i2}u + \lambda_{i3}v + \lambda_{i4}w = f_i \qquad (i = 1, 2), \tag{5}$$

where $\lambda_{ij}(x)$ and $f_i(x)$ $(i = 1, 2;\ j = 1, 2, 3, 4)$ are real functions Hölder continuous on S.

Consider the exterior Dirichlet problem:

Find a vector $V[U, W]$, holomorphic outside the ball G, vanishing at infinity, Hölder continuous up to S, and satisfying on S the condition

$$U\big|_S = f(x), \qquad f(x) \in W_2^{(2)}(S).$$

This problem is, generally speaking, unsolvable; however, the modified Dirichlet problem

$$U\big|_S = f(x) + \overline{\Phi(\zeta)},$$

where $\Phi(\zeta)$ is an analytic function of ζ, not prescribed in advance but determined entirely by the function $f(x)$, is always solvable.

Theorem. *Every vector holomorphic in the ball G, whose boundary values belong to the class $W_2^{(2)}(S)$, admits the representation*

$$V(x) = \frac{1}{2\pi} \iint_S D' \frac{1}{|x - \xi|}\, DS_\xi \cdot \mu(\xi) + V_0(\zeta), \tag{6}$$

where

$$DS_\xi = \sum_{i=1}^{3} \gamma_i \xi_i \, dS_\xi;$$

$\mu(x)$ is a real vector with components $(\mu_1, 0, 0, \mu_4)$, Hölder continuous on S; $V_0(\zeta)$ is a real vector with components $(0, \Phi_2, \Phi_3, 0)$, where $\Phi(\zeta) = \Phi_2(\zeta) - i\Phi_3(\zeta)$ is an analytic function of ζ. Moreover, the analytic function $\Phi(\zeta)$ and the components μ_1 and μ_4 are uniquely determined by the vector $V(x)$.

We shall carry out the proof analogously to the planar case (see (^{3}), Chap. III). Let $V[U, W]$ be a vector holomorphic in the ball G. It can be represented by means of an integral of Cauchy type (see (^{1,2})):

$$V(x) = \frac{1}{4\pi} \iint_S D' \frac{1}{|x-\xi|} DS_\xi \cdot V(\xi). \tag{7}$$

Let $V_1[U_1, W_1]$ be a solution of the modified Dirichlet problem for the exterior of the ball G:

$$U_1^-\big|_S = U^+\big|_S - \overline{\Phi(\zeta)}.$$

Then for every point $x \in G$

$$\frac{1}{4\pi} \iint_S D' \frac{1}{|x-\xi|} DS_\xi \cdot V_1^-(\xi) = 0. \tag{8}$$

Put

$$V^+(x) - V_1^-(x) = 2\mu(x) + V_0(\zeta),$$

where $V_0(0, \Phi_2(\zeta), \Phi_3(\zeta), 0)$, with $\Phi(\zeta) = \Phi_2 - i\Phi_3$ an analytic function of ζ.

Subtracting formulas (7) and (8), we obtain

$$V(x) = \frac{1}{2\pi} \iint_S D' \frac{1}{|x-\xi|} DS_\xi \cdot \mu(\xi) + \frac{1}{4\pi} \iint_S D' \frac{1}{|x-\xi|} DS_\xi \cdot V_0(\xi).$$

Since $V_0(\zeta)$ is a vector holomorphic in the ball G, the last term is equal to $V_0(\zeta)$, and we obtain (6).

The integral representation (6) makes it possible to reduce the boundary-value problem with the general boundary conditions (5) to a system of singular integral equations.

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Note: Figure translations are in progress. See original paper for figures.

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