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Abstract

Full Text

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ON INFINITE GROUPS OF MAPPINGS DEFINED BY DIFFERENTIAL EQUATIONS

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The usual geometric interpretation of solutions of a system of first-order partial differential equations for functions u, v of independent variables x, y

$$F_i(x, y, u, v, u_x, u_y, v_x, v_y) = 0 \qquad (i = 1, 2) \tag{1}$$

consists in regarding each of its solutions as a mapping $(x, y) \to (u, v)$. In this case system (1) describes certain geometric properties of the principal linear part of such a mapping at any given point (x, y). In this interpretation the set of all solutions of the given system (1) forms a class Γ of mappings. It does not follow at all that the superposition of two mappings from the given class Γ itself belongs to Γ. Sometimes this property is satisfied: for example, it holds for the Cauchy-Riemann system, when Γ is the class of conformal mappings. Moreover, in this particular case Γ is a group. In this connection it is of interest to find all such systems (1) for which the class Γ is a group. The present paper is devoted to the solution of this problem.

If Γ is a group, then Γ must contain the identity mapping. This means that (1) has the solution $u = x,\ v = y$. Therefore the "initial conditions"

$$F_i(x, y, x, y, 1, 0, 0, 1) = 0 \qquad (i = 1, 2). \tag{2}$$

must be satisfied.

We now show that if the "initial conditions" (2) are satisfied for (1), then the requirement that Γ be a group is equivalent to the requirement that system (1) admit the group Γ in the sense of Lie ([1]). Indeed, suppose that (1) admits the transformation

$$T: \quad x' = u_0(x, y), \quad y' = v_0(x, y).$$

This means that the system (1′), obtained by transforming (1) to the independent variables x', y', coincides with (1). Therefore $u = x',\ v = y'$ is a solution of (1′), and hence the functions $u_0(x, y), v_0(x, y)$ form a solution of (1). Consequently, $T \in \Gamma$. Conversely, if Γ is a group, then the superposition $T'T$ of any two solutions of (1) is again a solution of (1). On the other hand, this superposition is a solution of a certain system ($1'$) obtained from (1) by the transformation T. Thus T' is a solution of ($1'$), and since T' was an arbitrary solution of (1), the two systems (1) and ($1'$) have the same solutions, i.e., they coincide. Consequently, system (1) admits any transformation $T \in \Gamma$.

Further, as a result of a smooth locally homeomorphic transformation of the variables x, y, system (1)

$$\Pi: \quad \bar{x} = \varphi(x, y), \quad \bar{y} = \psi(x, y)$$

and by the same transformation of the variables u, v

$$\Pi: \quad \bar{u} = \varphi(u, v), \quad \bar{v} = \psi(u, v)$$

passes into some system ($\bar{1}$), for which the class $\bar{\Gamma}$ will be described by the formula

$$\bar{\Gamma} = \Pi \Gamma \Pi^{-1}. \tag{3}$$

Consequently, if Γ is a group, then $\bar{\Gamma}$ is also a group. Thus, from each system (1) for which Γ is a group, one can obtain as many new systems as desired with the same property. In connection with this, it is expedient to adopt the following definition: two systems (1) and ($\bar{1}$), whose classes Γ and $\bar{\Gamma}$ are related by formula (3), are called equivalent.

By virtue of this definition, the totality of all possible systems (1) decomposes into classes of equivalent systems. It is now clear that it suffices to solve the problem posed up to equivalence, i.e., to find the simplest representatives of each class.

Let us first consider those systems (1) in whose left-hand sides the variables x, y, u, v do not enter explicitly and which are reducible to the Cauchy normal form:

$$u_x = f(u_y, v_y), \quad v_x = g(u_y, v_y). \tag{4}$$

For these systems the "initial conditions" (2) take the form

$$f(0, 1) = 1, \quad g(0, 1) = 0. \tag{5}$$

In what follows we use the notation: $u_y = \alpha,\ v_y = \beta$. For these quantities the "initial conditions" are

$$\alpha = 0, \quad \beta = 1. \tag{6}$$

We shall call the totality of conditions (5) and (6) the "initial point." Let the class Γ for the system (4) be a group, and let

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

be the operator of some one-parameter subgroup of Γ. By the usual construction ([2]) we obtain the following defining equations, which express the fact of invariance of the system (4) with respect to transformations with operator X:

$$f\xi_x + \alpha\eta_x = (ff_\alpha + gf_\beta)\xi_y + (\alpha f_\alpha + \beta f_\beta)\eta_y,$$

$$g\xi_x + \beta\eta_x = (fg_\alpha + gg_\beta)\xi_y + (\alpha g_\alpha + \beta g_\beta)\eta_y, \tag{7}$$

where the indices denote the corresponding partial derivatives. Equations (7) must be satisfied identically in the independent variables x, y, α, β.

In particular, at the "initial point" we have

$$\xi_x = p\xi_y + q\eta_y, \quad \eta_x = r\xi_x + s\eta_y, \tag{8}$$

where the following notation has been used:

$$f_\alpha(0,1) = p, \quad f_\beta(0,1) = q, \quad g_\alpha(0,1) = r, \quad g_\beta(0,1) = s. \tag{9}$$

For what follows it is necessary to have more detailed information about the class Γ. Suppose that, for the system (4), the Cauchy problem with "arbitrary" initial data at $x = 0$ is solvable. Each such

a solution "sufficiently close" to the identity can be included in a one-parameter subgroup with some operator X. Therefore the functional arbitrariness in the definition of the operator X must be no less than the arbitrariness in the definition of the solution of system (4). This means that no conditions may be imposed on the functions ξ, η beyond the conditions (8). Hence it follows, in particular, that the determining equations (7) must be satisfied by virtue of (8) identically not only with respect to the variables x, y, α, β, but also with respect to the variables $\xi_x, \xi_y, \eta_x, \eta_y$. In this case Γ will be an infinite group. Therefore the preceding conclusion may be called the condition for the infiniteness of the group Γ.

Substitution of ξ_x, η_x from (8) into (7) and use of the condition for the infiniteness of Γ leads to the equations

$$\left.\begin{aligned} ff_\alpha + gf_\beta = pf + r\alpha, \qquad fg_\alpha + gg_\beta = pg + r\beta,\\ \alpha f_\alpha + \beta f_\beta = qf + s\alpha; \qquad \alpha g_\alpha + \beta g_\beta = qg + s\beta. \end{aligned}\right\} \tag{10}$$

Computation shows that the integrability conditions of system (10) reduce to the following:

$$(q-1)p + (q+1)s = 0, \qquad 2(q-1)r + (s-p)s = 0. \tag{11}$$

When conditions (11) are fulfilled, system (10) will be completely integrable.

To solve the problem up to equivalence, it remains only to determine to what simplest forms the vector (p, q, r, s) can be reduced by linear equivalence transformations

$$\bar{x} = ax + by, \qquad \bar{y} = cx + dy;$$

$$\bar{u} = au + bv, \qquad \bar{v} = cu + dv, \tag{12}$$

where a, b, c, d are parameters of the transformation. These transformations preserve the form of system (4). Computing the prolongation of the transformations (12) to the derivatives u_x, u_y, v_x, v_y, we obtain the induced transformations of the variables α, β, f, g. Prolonging once more the transformations obtained to the derivatives of f, g with respect to α, β and considering the result at the "initial point," we finally obtain the following operators of the group of linear equivalence transformations:

$$Y_1 = p\frac{\partial}{\partial p} + 2r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}, \qquad Y_2 = (q+1)\frac{\partial}{\partial p} + (s-p)\frac{\partial}{\partial r} - (q-1)\frac{\partial}{\partial s},$$

$$Y_3 = (p^2+qr+r)\frac{\partial}{\partial p} + (pq-p+qs+s)\frac{\partial}{\partial q} + r(p+s)\frac{\partial}{\partial r} + (qr-r+s^2)\frac{\partial}{\partial s}. \tag{13}$$

The manifold defined by equations (11) is invariant with respect to the group (13).

Consider the alternative: either $q = 1$, or $q \neq 1$.

If $q = 1$, then from (11) we obtain $s = 0$, and conditions (11) will be satisfied. Further, by a transformation with operator Y_2, preserving the values $q = 1,\ s = 0$, one can make $p = 0$. After this, by a transformation with operator Y_1, preserving the values $q = 1,\ s = 0,\ p = 0$, one can make $r = \pm 1$ (for $r \neq 0$).

If, however, $q \neq 1$, then by a transformation with operator Y_2 one can make $s = 0$, and then from (11) we obtain $p = r = 0$.

Thus any vector (p, q, r, s) satisfying (11) is equivalent to one of the following vectors: $(0, 1, -1, 0)$; $(0, 1, 1, 0)$; $(0, q, 0, 0)$. Under the "initial conditions" (5), (6), for each of the corresponding vectors the system (10) has a unique solution, namely

$$\begin{aligned} (0,1,-1,0):& \qquad f = \beta, \quad g = -\alpha; \\ (0,1,1,0):& \qquad f = \beta, \quad g = \alpha; \\ (0,q,0,0):& \qquad f = (\beta)^q, \quad g = 0. \end{aligned}$$

This proves that every system of the form (4) defining an infinite group of transformations Γ is equivalent to one of the following systems:

$$u_x = v_y, \qquad v_x = -u_y; \tag{14a}$$

$$u_x = v_y, \qquad v_x = u_y; \tag{14b}$$

$$u_x = (v_y)^q, \qquad v_x = 0. \tag{14c}$$

In system (14c), q is an arbitrary real number.

An analogous analysis shows that system (1), still not explicitly containing the variables x, y, u, v, but not reducible to Cauchy normal form, is equivalent to one of the systems

$$u_x = pv_x + 1, \qquad u_y = pv_y - p, \tag{15}$$

where p is an arbitrary constant.

For systems (1) of general form the analysis becomes more complicated; nevertheless here, too, it is possible to show that any such system defining an infinite group of transformations is equivalent to one of the systems (14) or (15). In doing so, of course, the general equivalence transformations (3) are considered.

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Note: Figure translations are in progress. See original paper for figures.