ON FINITE GROUPS WITH Π-DECOMPOSABLE SUBGROUPS

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Abstract

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ON FINITE GROUPS WITH Π-DECOMPOSABLE SUBGROUPS

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§ 1. In papers ([1–4]) S. A. Chunikhin defined classes of groups broader than soluble and supersoluble groups: Π-soluble, Π-separable, and strongly Π-soluble groups. S. A. Chunikhin for Π-soluble and Π-separable groups ([1–2]), and P. Hall for Π-separable groups ([5]), showed that for these groups theorems analogous to P. Hall' s theorem for soluble groups ([6]) hold.

In recent years many authors ([7–14]) have investigated the question of solubility or supersolubility of a group $G = AB$, depending on the properties of the subgroups A and B. In the present paper a similar question is considered for the properties of Π-separability, Π-solubility, and strong Π-solubility of G (see Theorems 2, 4–7, 9).

From the example of the simple group of order 60 it is clear that a complete analogue of the theorems of papers ([7, 11–14]) for the properties of Π-solubility or strong Π-solubility of G, replacing nilpotency of A and B by Π-decomposability, cannot be obtained. But the paper shows that, with some violation of the analogy, similar theorems are true (see Theorems 4–7, 9).

In Theorems 1, 3, and 8, from the property of a maximal subgroup of the group G, a property of the group G is determined. In Theorem 8 a criterion is given for strong Π-separability introduced by S. A. Rusakov ([15]).

Theorem 10 weakens the condition of P. Hall' s theorem (([16]), Theorem 10.5.7), while Theorem 11 generalizes Theorem 9 of Huppert' s paper ([17]).

Theorems of papers ([18, 19]) of Carter are special cases of Theorems 13 and 15. Theorems 14 and 16 give a characterization of the subgroups found by Carter ([18]) in soluble groups.

§ 2. We give the definitions and notation used in the paper: Π is a certain set of primes, and Π' is its complement; G is a finite group; (B) is the order of the group B; $\Pi(G)$ is the set of all prime divisors of (G); a subgroup whose order is equal to the greatest Π-Sylow divisor of (G) (see ([20])) will be called, following Wielandt ([21]), a Π-Hall subgroup of the group G; the identity subgroup E of the group G will be regarded as a p-Sylow subgroup for any prime number p not dividing (G); by a maximal subgroup of the group G we mean, for $G \neq E$, such a proper subgroup which is not a proper subgroup of any proper subgroup of the group G, and for $G = E$, the group G itself; a subgroup $\mathfrak{H}$ will be called an abnormal subgroup ([18]) of the group G if $a \in \{\mathfrak{H},\ a^{-1}\mathfrak{H}a\}$ for every element $a \in G$.

Analogously to Definition 2 of paper ([3]), a group G will be called Π-decomposable if it decomposes into the direct product of two sets, one of which is its nilpotent Π-Hall subgroup.

By a K_π-subgroup of the group G we shall mean a Π-decomposable subgroup which coincides with its normalizer in G, and whose order is divisible by the greatest Π'-Sylow divisor of (G). If $\Pi = \Pi(G)$, then a K_π-subgroup of the group G will be called a K-subgroup. It is obvious that a K-subgroup of the group G is a nilpotent subgroup coinciding with its normalizer in G.

The definition of regular Sylow subgroups is introduced in paper ([22]).

§ 3. **Lemma.** Let $\mathscr{H}$ be a non-Sylow Π-Hall subgroup of a group G, contained in some Π-decomposable subgroup A. If, for every $p \in \Pi$, a Sylow p-subgroup of $\mathscr{H}$ has as its normalizer in G the subgroup A, then G has an invariant complement to $\mathscr{H}$.

§ 4. **Theorem 1.** If a maximal subgroup M of a group G is a Π-decomposable subgroup whose index is not divisible by any number from Π, or is a power of some prime in Π, then G is a Π-separable group.

Theorem 2. Let $G = AB$, where A and B are maximal and Π-decomposable subgroups of G.

Then G is a Π-separable group.

§ 5. **Theorem 3.** Let a maximal subgroup M of a group G be Π-decomposable, and suppose that, in the case when $2 \in \Pi$, a Sylow 2-subgroup of M is either invariant in G or regular. If the index of M is divisible by some power of only one number from Π, then G is Π-separable; and if the index of M is not divisible by any number from Π, then G is a Π-solvable group.

Theorem 4. Let $G = AB$, where A and B are maximal and Π-decomposable subgroups of G. If, in the case when $2 \in \Pi$, each of the Sylow 2-subgroups of A and B is either invariant in G or regular, then G is Π-solvable.

Theorem 5. Let $G = AB$, where A is a Π-decomposable subgroup and the subgroup B is abelian or Hamiltonian. If (B) is divisible by some power of only one number from Π, then G is Π-separable; and if (B) is not divisible by any number from Π, then G is Π-solvable.

Theorem 6. Let $G = AP$, where A is a Π-decomposable subgroup and P is a p-subgroup, where p is a prime. If $p \in \Pi$, then G is Π-separable; and if $p \notin \Pi$ and A is p-solvable, then G is a Π-solvable group.

Theorem 7. Let $G = AB$, where A is a Π-decomposable subgroup of odd order, and the subgroup B has a subgroup $\mathfrak{L}$ of index 2 such that every subgroup of $\mathfrak{L}$ is a normal divisor of B. If (B) is divisible by some power of only one number from Π, then G is Π-separable; and if (B) is not divisible by any number from Π, then G is a Π-solvable group.

§ 6. **Theorem 8.** Let a maximal subgroup M of a group G be a Π-decomposable subgroup with cyclic Sylow p-subgroups for all $p \in \Pi$. If the index of M is divisible only by the first power of just one prime from Π, then G is strongly Π-separable; and if the index of M is not divisible by any prime from Π, then G is strongly Π-solvable.

Theorem 9. Let $G = AB$, where A and B are maximal and Π-decomposable subgroups of G with cyclic Sylow p-subgroups for all $p \in \Pi$. Then G is strongly Π-solvable.

§ 7. In the example of the simple group of order $60 = 5 \cdot 12$, which is the product of two $\{5\}$-decomposable subgroups, of which the subgroup of order 12 is maximal, it is clear that in Theorems 4 and 9 the condition of maximality for one of the subgroups A and B cannot be omitted. This example also shows that the condition on the index of the maximal subgroup in Theorems 3 and 8 cannot be omitted.

§ 8. **Theorem 10.** Let P be a Sylow p-subgroup of G, where p is the greatest number in $\Pi(G)$. If all maximal subgroups of G containing P have as their index a prime number or the square of a prime number, then G is solvable.

Theorem 11. Let Π be a finite set of primes containing $\Pi(G)$, and let P be a cyclic Sylow p-subgroup of G, where p is the greatest number in Π. The group G is strongly solvable if and only if all maximal subgroups of G containing P have prime index.

Theorem 12. Let P be a Sylow p-subgroup of G and let q be the smallest number in $\Pi(G)$. If the index of any maximal subgroup of G containing P is equal to q or q^2, then G is a solvable group of order $p^\alpha q^\beta$.

§ 9. **Theorem 13.** Let G contain a solvable Π-Hall normal divisor. Then G has at least one K_Π-subgroup, any two K_Π-subgroups of the group G are conjugate to one another, and every K_Π-subgroup of the group G is abnormal.

In a solvable group there always exists a K-subgroup [18], whose characteristic is given by the following

Theorem 14. Let T be a Hall normal divisor of a solvable group G, and let T' be a subgroup supplementary to T in G. Then every K-subgroup of T' is contained in a subgroup conjugate to the given K-subgroup of the group G.

§ 10. **Theorem 15.** Let G have a Π-Hall subgroup $\mathcal{H}$, each Sylow subgroup of which is either invariant in G or regular. If $\mathcal{H}$ is contained in some abnormal Π-decomposable subgroup of the group G, then G has an invariant complement to $\mathcal{H}$.

Theorem 16. Let a solvable group G have a Hall nilpotent subgroup $\mathcal{H}$, each Sylow subgroup of which is either invariant in G or regular. Then G has an invariant complement to $\mathcal{H}$ if and only if $\mathcal{H}$ is contained in some K-subgroup of the group G.

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Note: Figure translations are in progress. See original paper for figures.