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Abstract

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MATHEMATICS

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COLLAPSING FINITE SYSTEMS OF LINEAR INEQUALITIES

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For an arbitrary system

$$f_j(x) + t_j \leq 0 \qquad (j = 1, 2, \dots, m), \tag{1}$$

in which $f_j(x)$ are real linear functions defined on some real linear space L, and t_j are free real parameters (a system of linear inequalities over L), and for an arbitrary subspace U of L, we define finite systems (let P be any one of them) satisfying the following two conditions.

a) Each inequality of the system P is such a linear combination of the inequalities of system (1) with nonnegative, but not all zero, coefficients (a positive combination) that the combination of functions $f_j(x)$ occurring in it is identically equal to zero on the chosen subspace U of L.

b) For every set of values of the parameters t_j, the set of solutions of the system P in some fixed direct complement V of the subspace U in L coincides with the projection onto V of the set of solutions of system (1).

For brevity, an arbitrary system P of this kind will be called a UV-combination of system (1). Some special UV-combinations of system (1) (its collapses) were considered earlier in the paper ([1]). In the present paper the basic properties of arbitrary UV-combinations of system (1) are established, and a description is given of the totality of all its UV-combinations. The study of UV-combinations is completed below by an algorithm for obtaining a certain (finite) sequence of them with rank decreasing to zero.

1. Let U be some subspace of L. Denote by $C(U)$ the cone of nonnegative solutions $(u_1, \dots, u_m)$ of the equation

$$u_1 f_1(x) + \dots + u_m f_m(x) = 0,$$

in which the functions $f_j(x)$ are considered only for $x \in U$. A nonzero element $(u_1^0, \dots, u_m^0)$ of the cone $C(U)$ will be called a fundamental element if the rank of the system of functions $f_j(x)$ $(x \in U)$, corresponding to its positive coordinates, is one less than their number. In this case the combination

$$u_1^0 f_1(x) + \dots + u_m^0 f_m(x)$$

will be called fundamental on U. It is not difficult to verify that the cone $C(U)$ can contain no more than a finite number of essentially distinct fundamental elements. Two elements of the cone $C(U)$ are not considered here essentially distinct if they differ from one another only by a positive numerical factor. The following assertion is also true:

Maximal systems of essentially distinct fundamental elements of the cone $C(U)$, and only they, are its bases.

A basis is an irreducible system of generating elements.

If $C_i = (u_1^{(i)}, \dots, u_m^{(i)})$ $(i = 1, 2, \dots, k)$ is some finite system of nonzero generating elements of the cone $C(U)$, then the system

$$\sum_{j=1}^{m} u_j^{(i)} f_j(x) + \sum_{j=1}^{m} u_j^{(i)} t_j \leq 0 \qquad (i = 1, 2, \dots, k)$$

will be called a U-collapse of system (1); if the cone $C(U)$ is the zero cone, then we say that the U-collapse of system (1) is empty. If C_i $(i = 1, 2, \dots, k)$ is a basis of the cone $C(U)$, then the U-collapse under consideration will be called fundamental.

In view of the preceding assertion, every nonempty U-convolution of system (1) contains, as a subsystem, a fundamental U-convolution of the latter.

Theorem 1. If the cone $C(U)$ is nonzero, then every U-convolution of system (1) is a UV-combination of the latter under any direct complement V of the subspace U in L, and, conversely, if some inequality over L is a UV-combination of system (1) for at least one $V = V^0$, then it coincides with some U-convolution of system (1).

It follows from this that the totality of UV-combinations with one and the same U does not depend on the choice of V and is determined one-to-one by the various finite systems of nonzero generating elements of the cone $C(U)$.

Theorem 2. System (1) with a nonempty U-convolution S for some $U \subset L$ has solutions in L for those and only those values of the parameters t_j for which its U-convolution S has solutions in L. System (1) with an empty U-convolution for some $U \subset L$ has solutions in U for all values of the parameters entering into it.

2. Let U and U' be any two subspaces of L. If the U-convolution S of system (1) is nonempty, then the U'-convolution of the system S (considered over L) will be called the repeated $(U; U')$-convolution of system (1); if the U-convolution S is empty, or if the U'-convolution of the system S is empty, then we regard the $(U; U')$-convolution of system (1) as empty. The repeated $(U; U'; U'')$-convolution of system (1), etc., is defined analogously.

With the aid of Theorem 1 one can prove that any $(U; U')$-convolution of system (1) coincides with some $(U + U')$-convolution of the latter.

If S is some nonempty U-convolution of system (1), then, in view of the assertion preceding Theorem 1, it follows that if, when forming the U'-convolution of the system S, one does not take those of the inequalities thereby obtained into which there do not enter the combinations, fundamental on $U + U'$, of the functions $f_j(x)$ of system (1), then the system thus obtained will be a $(U+U')$-convolution of system (1). This proposition is used essentially in the algorithm of reduced convolution proposed below for the system

$$f_j(x) + t_j \equiv a_{j1}x_1 + \dots + a_{jn}x_n + t_j \leqslant 0 \quad (j = 1, 2, \dots, m) \tag{2}$$

over the subspace R^n, which yields successively its U-convolutions for $U = U_1$, $U_1 + U_2, \dots, U_1 + U_2 + \dots + U_n$, where $U_1, U_2, \dots, U_n$ are the subspaces generated in R^n, respectively, by the coordinate vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$.

Before passing to the exposition of this algorithm, let us give the following definition. Let A be some system of m real numbers $a_1, a_2, \dots, a_m$, and let Z be a system of m unknowns $z_1, z_2, \dots, z_m$. To each pair of elements $a_p > 0$ and $a_q < 0$ from A we assign the linear form $a_p z_q - a_q z_p$, and to each zero element a_s from A the form z_s; we shall say that the totality of all forms obtained in this way is obtained as the result of the A-deformation of the system Z. If in the system A there occur numbers of opposite signs, then the A-deformation will be called a weaving deformation.

Algorithm of reduced convolution

1. Let A_1 be some nonzero column of coefficients of system (2), for example, the column of coefficients of the unknown x_1. Performing the A_1-deformation of the left-hand sides of the inequalities of system (2), we obtain a system S_1, which is its U_1-convolution (obviously fundamental). To each inequality of the system S_1 we assign its index—the set of numbers of those inequalities of system (2) by whose combination it was obtained.

2. Suppose that a system S_k has already been obtained which is a fundamental $(U_1 + \dots + U_k)$-convolution of system (2), and that A_{k+1} is some nonzero column of its coefficients, for example, the column of coefficients of x_{k+1}. If the A_{k+1}-deformation is not a weaving deformation, then the system S_{k+1}

is obtained by performing the A_{k+1}-deformation of the left-hand sides of the inequalities of the system S_k.

If, however, it is a deformation of a convolution and if among the A_i-deformations with $i = 1, 2, \dots, k$ there were s deformations that were deformations of a convolution, then in order to obtain the system S_{k+1} we proceed as follows:

1) we subject the left-hand sides of the inequalities of the system S_k to an A_{k+1}-deformation, but in doing so we do not combine the left-hand sides of those inequalities of the system S_k whose union of indices contains more than $s + 1$ distinct elements;

2) from the system thus obtained we exclude, one after another, every inequality whose index contains the index of at least one of the inequalities remaining in it.

In both cases the system S_{k+1} is a fundamental $(U_1 + \dots + U_{k+1})$-convolution of the system (2). As above, to each inequality of the system obtained we assign its index.

Continuing this process and regarding any convolution of an empty convolution as empty, we obtain, after a finite number of steps (not exceeding the rank of the system (2)), an R^n-convolution of the system (2).

Remark. If the A_i-deformation is carried out here without the restrictions connected with taking account of the indices of the inequalities obtained, then the algorithm considered turns into the algorithm of successive convolution from article (1).

If the algorithm considered is applied to the system (2) for certain fixed values of the parameters t_j, then from the consistency (inconsistency) in R^n of any of the convolutions obtained in it there will follow, by virtue of Theorem 2, the consistency (inconsistency) of the system (2) under consideration; moreover, each solution of any of them can obviously be successively extended to some solution of the latter.

Example. Find some solution of the system

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 - 3 &\leqslant 0, \\ 2x_1 - x_2 - x_3 - x_4 + 1 &\leqslant 0, \\ -x_1 + 2x_2 + x_3 - x_4 - 2 &\leqslant 0, \\ -x_1 - x_2 + 2x_3 + x_4 - 2 &\leqslant 0, \\ 3x_1 + x_2 - 3x_3 - 2x_4 + 1 &\leqslant 0, \\ -2x_1 - x_2 + x_3 + x_4 + 0 &\leqslant 0. \end{aligned}$$

For it the U_1-convolution has the form

$$
\begin{array}{rcll}
3x_2+0x_3+0x_4-5 & \leqslant & 0, & (1;3) \\
0x_2+x_3+2x_4-5 & \leqslant & 0, & (1;4) \\
x_2-x_3+3x_4-6 & \leqslant & 0, & (1;6) \\
3x_2+x_3-3x_4-3 & \leqslant & 0, & (2;3) \\
-3x_2+3x_3+x_4-3 & \leqslant & 0, & (2;4)
\end{array}
\qquad
\begin{array}{rcll}
-2x_2+0x_3+0x_4+1 & \leqslant & 0, & (2;6) \\
7x_2+0x_3-5x_4-5 & \leqslant & 0, & (3;5) \\
-2x_2+3x_3+x_4-5 & \leqslant & 0, & (4;5) \\
-x_2-3x_3-x_4+2 & \leqslant & 0, & (5;6).
\end{array}
$$

Next to the inequalities their indices are written.

We next compose the (U_1+U_3)-convolution

$$
\begin{array}{rcll}
3x_2+0x_4-5 & \leqslant & 0, & (1;3) \\
-2x_2+0x_4+1 & \leqslant & 0, & (2;6) \\
7x_2-5x_4-5 & \leqslant & 0, & (3;5)
\end{array}
\qquad
\begin{array}{rcll}
x_2+5x_4-11 & \leqslant & 0, & (1;4;6) \\
-3x_2+0x_4-3 & \leqslant & 0, & (4;5;6).
\end{array}
$$

We then take the $(U_1+U_3+U_4)$-convolution

$$
\begin{array}{rl}
3x_2-5\leqslant 0, & (1;3) \\
-2x_2+1\leqslant 0, & (2;6) \\
-3x_2-3\leqslant 0, & (4;5;6).
\end{array}
$$

Substituting one of the solutions of the latter, for example $x_2=1$, into the preceding convolution, we take $x_4=1$. Substituting $x_2=x_4=1$ into the U_1-convolution, we take $x_3=0$. Finally, substituting $x_2=x_4=1$ and $x_3=0$ into the original system, we find the solution $(0,1,0,1)$ of the latter.

Let us now consider the question of finding a general formula for the solutions of the system

$$a_{j1}x_1+\dots+a_{jn}x_n-a_j\leqslant 0 \quad (j=1,2,\dots,m) \tag{3}$$

of rank $r>0$.

First note the following rule, which follows directly from the results of Sec. 1.

To obtain the general form of the nonnegative solutions of the system of homogeneous linear equations

$$a_{1i}u_1+\dots+a_{mi}u_m=0 \quad (i=1,2,\dots,n) \tag{4}$$

one should construct the fundamental R^n-convolution of the system of inequalities associated with it,

$$a_{j1}x_1+\dots+a_{jn}x_n+t_j\leqslant 0 \quad (j=1,2,\dots,m).$$

If

$$u_1^{(k)}t_1 + \dots + u_m^{(k)}t_m \leqslant 0 \quad (k = 1, 2, \dots, l)$$

are the inequalities of the convolution thus constructed, then $(u_1^{(k)}, \dots, u_m^{(k)})$ $(k = 1, 2, \dots, l)$ is a maximal system of essentially distinct fundamental elements of the cone of nonnegative solutions of the system (4), and

$$(u_1, \dots, u_m) = \sum_{k=1}^{l} p_k(u_1^{(k)}, \dots, u_m^{(k)}) \tag{5}$$

is the formula which, for all possible nonnegative values of the parameters p_k, gives all nonnegative solutions of the system (4).

The system (3) with $a_j = 0$ $(j = 1, 2, \dots, m)$ is equivalent to the system of equations

$$a_{j1}x_1 + \dots + a_{jn}x_n = -u_j \quad (j = 1, 2, \dots, m)$$

with nonnegative parameters u_j. Since $r > 0$, Gaussian elimination of the unknowns can be applied to the latter. Without loss of generality, one may assume that the resulting formulas for the unknowns and equations for the parameters u_j have, respectively, the form

$$x_s = b_{s1}u_1 + \dots + b_{sr}u_r + a'_{s1}x_{r+1} + \dots + a'_{sn-r}x_n \quad (s = 1, 2, \dots, n-r); \tag{6}$$

$$c_{i1}u_1 + \dots + c_{ri}u_r + u_{r+i} = 0 \quad (i = 1, 2, \dots, m-r). \tag{7}$$

If (5) is an expression for an arbitrary solution of the system (7), then, substituting it into the formulas (6), we obtain the general formula for the solutions of the system (3) in the case $a_j = 0$ $(j = 1, 2, \dots, m)$; in these formulas x_{r+i} $(i = 1, 2, \dots, n-r)$ are free parameters.

The case of an arbitrary system (3) is reduced to the case considered, since it is equivalent to the system

$$a_{j1}x_1 + \dots + a_{jn}x_n - a_{jx_{n+1}} \leqslant 0,$$

$$x_{n+1} = 1.$$

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1. S. N. Chernikov, DAN, **131**, No. 3, 518 (1960).

Note: Figure translations are in progress. See original paper for figures.