ON THE EXTENSION OF FUNCTIONS

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Abstract

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MATHEMATICS

Ya. S. BUGROV

ON THE EXTENSION OF FUNCTIONS

(Presented by Academician S. L. Sobolev, 7 I 1963)

1. In this note we consider the question of extending functions defined on the subspace $R_{n-1} = \{-\infty < x_i < \infty,\ i = 1, \dots, n-1\}$ to the upper half-space $R_n^0 = \{x_n > 0,\ -\infty < x_i < \infty,\ i = 1, \dots, n-1\}$. In order to avoid cumbersome notation, we shall restrict ourselves to the case $n = 3$, although all the arguments carry over to the case of arbitrary n.

Definition. Let $r_i = \bar{r}_i + \alpha_i$, $\bar{r}_i \geq 0$ integers, $0 < \alpha_i \leq 1$ $(i = 1, 2, 3)$. A function $f(x, y, z)$ is said to **belong to the class** $S_p^{(r_1,r_2,r_3)} H(R_3) = S_p^{(r_1,r_2,r_3)} H$ $(1 \leq p \leq \infty)$ if: a) $f \in L_p(R_3)$ with norm

$$\|f\|_{L_p(R_3)} = \left(\int_{R_3} |f|^p \, dx \, dy \, dz \right)^{1/p} < \infty;$$

b) all possible generalized derivatives (in the sense of Sobolev) $\partial^{k_1+k_2+k_3} f / \partial x^{k_1} \partial y^{k_2} \partial z^{k_3}$ exist for $k_i = \bar{r}_i$ $(i = 1, 2, 3)$; c) for the derivatives indicated above the following relations hold:

$$\sup_h \left\| \frac{\Delta_h^2 \, \partial^{\bar{r}_1} f / \partial x^{\bar{r}_1}}{h^{\alpha_1}} \right\|_{L_p(R_3)} = M_p^{(r_1)}(f),$$

$$\dots$$

$$\sup_{h,k,l} \left\| \frac{\Delta_{h,k,l}^{2,2,2} \, \partial^{\bar{r}_1+\bar{r}_2+\bar{r}_3} f / \partial x^{\bar{r}_1} \partial y^{\bar{r}_2} \partial z^{\bar{r}_3}}{h^{\alpha_1} k^{\alpha_2} l^{\alpha_3}} \right\|_{L_p(R_3)} = M_p^{(r_1,r_2,r_3)}(f),$$

where

$$\Delta_h^2 \psi = \psi(x + 2h, y, z) - 2\psi(x + h, y, z) + \psi(x, y, z),$$

$$\Delta_{h,k}^{2,2} \psi = \Delta_h^2 [\Delta_k^2 \psi], \dots,$$

$$\Delta_{h,k,l}^{2,2,2} \psi = \Delta_h^2 [\Delta_k^2 (\Delta_l^2 \psi)]$$

(h is the increment in the variable x, k in the variable y, l in the variable z).

We introduce the norm in the space $S_p^{(r_1,r_2,r_3)} H$ as follows:

$$\|f\|_{S_p^{(r_1,r_2,r_3)} H} = \|f\|_{L_p(R_3)} + M_p^{(r_1)}(f) + \dots + M_p^{(r_1,r_2,r_3)}(f) < \infty.$$

It is clear from the definition that for $n = 1$ the class $S_p^{(r_1)} H \equiv H_p^{(r_1)}$ (for the definition of the class $H_p^{(r_1)}$, see (1a), p. 268).

The functional classes $S_p^{(r_1,r_2,r_3)}H$ were first introduced and studied by S. M. Nikol' skii. In the papers (1 ,) the classes $S_p^{(r_1,r_2,r_3)}W$ were also considered, where r_i $(i = 1, 2, 3)$ are integers.

One says that $f \in S_p^{(r_1,r_2,r_3)}W$ if $f \in L_p(R_3)$ and the generalized derivatives

$$\frac{\partial^{r_1} f}{\partial x^{r_1}}, \quad \frac{\partial^{r_2} f}{\partial y^{r_2}}, \quad \frac{\partial^{r_3} f}{\partial z^{r_3}}, \quad \frac{\partial^{r_1+r_2} f}{\partial x^{r_1}\partial y^{r_2}}, \dots, \quad \frac{\partial^{r_1+r_2+r_3} f}{\partial x^{r_1}\partial y^{r_2}\partial z^{r_3}}$$

are also integrable to the p-th power over R_3, and

$$\|f\|_{S_p^{(r_1,r_2,r_3)}W} = \|f\|_{L_p(R_3)} + \left\|\frac{\partial^{r_1} f}{\partial x^{r_1}}\right\|_{L_p(R_3)} + \dots + \left\|\frac{\partial^{r_1+r_2+r_3} f}{\partial x^{r_1}\partial y^{r_2}\partial z^{r_3}}\right\|_{L_p(R_3)} < \infty.$$

S. M. Nikol' skii proved (1) that the trace of a function f from the class $S_p^{(r_1,r_2,r_3)}H$ (for $z = 0$) belongs to the class $S_p^{(r_1,r_2)}H$. We shall prove the converse assertion–

namely, from the membership of the function $\varphi(x, y)$ in the class $S_p^{(r_1,r_2)}H$ it follows that it can be extended into the upper half-space in such a way that the extended function will belong to the class $S_p^{(r_1,r_2,r_3)}H(R_3^0)$ for any r_3.

2. Theorem 1. Let a system of functions $\varphi_0(x, y), \dots, \varphi_{s-1}(x, y)$ be given, belonging to the class $S_p^{(r_1,r_2)}H(R_2)$ $(1 \le p \le \infty,\ r_1 > 0,\ r_2 > 0)$. Then in the half-space R_3^0 one can construct a function $f(x, y, z)$ having the following properties:

a) $f \in S_p^{(r_1,r_2,r_3)}H(R_3^0)$ for any $r_3 > 0$, and the norm of f is estimated in terms of the norms of the functions φ_k $(k = 0, 1, \dots, s-1)$;

b) $\partial^k f(x, y, 0)/\partial z^k = \varphi_k(x, y)$ $(k = 0, 1, \dots, s-1)$.

(2,1)

Proof. We shall seek the function $f(x, y, z)$ in the form

$$f(x, y, z) = \exp(-z) \sum_{j=0}^{s_1-1} \left[\psi_j(x, y) \cos \beta_j z + \eta_j(x, y) \sin \beta_j z\right], \qquad (2,2)$$

where $\beta_i \ne \beta_j$ $(i \ne j)$ and $\beta_j > 0$ $(j = 0, 1, \dots, s_1 - 1)$ for $s = 2s_1$; $\beta_j > 0$ $(j = 1, 2, \dots, s_1 - 1)$, $\beta_0 = 0$ for $s = 2s_1 - 1$.

In view of the fact that $\beta_0 = 0$ when $s = 2s_1-1$, the function $\eta_0(x, y)$ may always be taken to be identically zero. Thus, for $s = 2s_1 - 1$ we have $2s_1 - 1$ unknown functions ψ_j, η_j. From formula (2,2), taking account of the equalities (2,1), in order to determine the functions ψ_j, η_j we obtain the system of equations:

$$\sum_{j=0}^{s_1-1}\psi_j=\varphi_0,$$

$$\sum_{j=0}^{s_1-1}\left[-\psi_j+\beta_j\eta_j\right]=\varphi_1,$$

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$$\sum_{j=0}^{s_1-1}\left[\left\{\binom{k}{0}-\binom{k}{2}\beta_j^2+\binom{k}{4}\beta_j^4-\cdots\right\}\psi_j+\left\{\binom{k}{1}\beta_j^{k-2}-\binom{k}{3}\beta_j^{k-4}+\cdots\right\}(-1)^{k_1}\beta_j\eta_j\right]=\varphi_k\quad(k=2k_1),\tag{2,3}$$

$$\sum_{j=0}^{s_1-1}\left[\left\{-\binom{k}{0}+\binom{k}{2}\beta_j^2-\cdots\right\}\psi_j+\left\{\binom{k}{k}\beta_j^{k-1}-\binom{k}{k-2}\beta_j^{k-3}+\cdots\right\}(-1)^{k_1}\beta_j\eta_j\right]=\varphi_k\quad(k=2k_1{+}1)$$

$$(k=0,1,\ldots,s-1).$$

Denote by Δ the determinant of the system (2,3), the j-th columns of which can be written as

$$1,\ -1,\ (1-\beta_j^2),\ldots$$

$$\ldots\left\{\begin{array}{ll}\left[\binom{k}{0}-\binom{k}{2}\beta_j^2+\binom{k}{4}\beta_j^4-\cdots\right] & \text{for } k=2k_1,\\ \left[-\binom{k}{0}+\binom{k}{2}\beta_j^2-\cdots\right] & \text{for } k=2k_1+1\end{array}\right\}\ldots$$

for $j\leq s_1-1$;

$$0,\ \beta_{j-s_1},\ -2\beta_{j-s_1},\ldots$$

$$\ldots\left\{\begin{array}{ll}(-1)^{k_1}\beta_{j-s_1}\left[\binom{k}{1}\beta_{j-s_1}^{k-2}-\binom{k}{3}\beta_{j-s_1}^{k-4}+\cdots\right] & \text{for } k=2k_1,\\ (-1)^{k_1}\beta_{j-s_1}\left[\binom{k}{k}\beta_{j-s_1}^{k-1}-\binom{k}{k-2}\beta_{j-s_1}^{k-3}+\cdots\right] & \text{for } k=2k_1+1\end{array}\right\}\ldots$$

for $s_1\leq j\leq 2s_1-1$.

In order that $\Delta \neq 0$, it is necessary that all β_i be distinct. Choosing the numbers β_i so that $\Delta \neq 0$ $(\beta_i \neq \beta_j,\ i \neq j)$, we obtain a solution of the posed problem.

In the general case it is obvious that

$$f(x,y,z) = \exp(-z)\sum_{i=0}^{s-1}\psi_i(z)\varphi_i(x,y), \tag{2.4}$$

where $\psi_i(z)$ are bounded functions and are a linear combination of the trigonometric functions $\cos\beta_i z,\ \sin\beta_i z$ with coefficients that ensure the fulfillment of conditions (2.1).

Let us give the solution of system (2.3) in the simplest cases:

1) $s = 1,\ f(x,y,z) = \exp(-z)\varphi_0(x,y)$;

2) $s = 2,\ f(x,y,z) = \exp(-z)[(\cos z + \sin z)\varphi_0(x,y) + \varphi_1(x,y)\sin z]$. (2.5)

Here we have put $\beta_0 = 1$.

3) $s = 3$ $(\beta_0 = 0,\ \beta_1 = 1)$,

$$f(x,y,z) = \exp(-z)[(2-\cos z+\sin z)\varphi_0(x,y)+$$

$$+(2-2\cos z+\sin z)\varphi_1(x,y) + (1-\cos z)\varphi_2(x,y)].$$

Now from formula (2.4) it is clear that, with respect to the variable z, the function $f(x,y,z)$ is infinitely differentiable. Therefore, if $\varphi_k(x,y) \in S_p^{(r_1,r_2)}H(R_2)$ $(k = 0,1,\dots,s-1)$, then $f \in S_p^{(r_1,r_2,r_3)}H(R_3^0)$ for any $r_3 > 0$. Conditions (2.1), generally speaking, are satisfied in the sense of convergence in the p-mean. Let us show this for the function (2.5). We have

$$\left(\iint_{R_2}|f(x,y,z)-\varphi_0(x,y)|^p\,dx\,dy\right)^{1/p} = \left(\iint_{R_2}|[\exp(-z)(\cos z+\sin z)-1]\times\right.$$

$$\left.\times\varphi_0(x,y)+\exp(-z)\sin z\cdot\varphi_1(x,y)|^p\ dx\,dy\right)^{1/p} \le cz\|\varphi_1\|_{L_p(R_2)}+$$

$$+c\left(\iint_{R_2}|\exp(-z)(\cos z-1)+\exp(-z)-1|^p|\varphi_0|^p\,dx\,dy\right)^{1/p}+$$

$$+cz\|\varphi_0\|_{L_p(R_2)} \le cz\left(\|\varphi_0\|_{L_p(R_2)}+\|\varphi_1\|_{L_p(R_2)}\right) \to 0 \quad \text{as } z \to 0.$$

Further, since

$$\frac{\partial f}{\partial z} = \exp(-z)[-2\sin z \cdot \varphi_0(x,y) + (\cos z - \sin z)\varphi_1(x,y)],$$

then, as above,

$$\left(\iint_{R_2} \left| \frac{\partial f}{\partial z} - \varphi_1 \right|^p dx\,dy \right)^{1/p} \leq cz \left(\|\varphi_0\|_{L_p(R_2)} + \|\varphi_1\|_{L_p(R_2)} \right) \to 0$$

as $z \to 0$, and the theorem is proved.

Remark 1. The theorem is also true in terms of $S_p^{(r_1,r_2,r_3)}W$.

Remark 2. In the case of extension to the whole space R_3, it is necessary in formula (2.2), instead of the factor $\exp(-z)$, to take the factor $\exp(-z^2)$. For example, for $s = 3$

$$f(x,y,z) = \exp(-z^2)[(3 - 2\cos z)\varphi_0(x,y) + (1 - \cos z)\varphi_2(x,y) +$$

$$+ \sin z \cdot \varphi_1(x,y)].$$

3. Let us show that for $n = 2,\ p = 2$ the corresponding extension is carried out in the form of a solution of a certain differential equation. Let

$$L^{(r,s)}u \equiv (-1)^{r+1}\frac{\partial^{2r}u}{\partial x^{2r}} + (-1)^{s+1}\frac{\partial^{2s}u}{\partial y^{2s}} + (-1)^{s+r+1}\frac{\partial^{2s+2r}u}{\partial x^{2r}\partial y^{2s}} = 0, \qquad (3.1)$$

where r, s are natural numbers (see (1^6)).

We pose the following problem for equation (3,1): to find in the upper half-plane $R_2^0 = \{y > 0,\ -\infty < x < \infty\}$ a bounded solution of equation (3,1) under the condition that

$$\frac{\partial^k u(x,0)}{\partial y^k} = \varphi_k(x) \quad (k = 0, 1, \dots, s-1), \qquad \lim_{y \to +\infty} u(x,y) = 0. \qquad (3,2)$$

We note that conditions (3,2) are understood in the sense of mean-square convergence. Applying the Fourier method of separation of variables, the solution of problem (3,1)–(3,2) (for $s = 1$) can be written in the form

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_0(\lambda) \exp(-\varkappa y) e^{i\lambda x}\, d\lambda,$$

where Φ_0 is the Fourier transform of the function $\varphi_0(x)$, $\varkappa = |\lambda|^r(1+\lambda^{2r})^{-1/2}$.

Theorem 2. If the function $\varphi_0(x) \in S_2^{(r_1)}H \equiv H_2^{(r_1)}(R_1)$, $(r_1 > 0)$, then the solution of the boundary-value problem (3,1)–(3,2) for $s = 1$ belongs to the class $S_2^{(r_1,r_2)}H[R_2(0,1)]$ for any $r_2 > 0$, and

$$\int_{-\infty}^{\infty} |u(x,y) - \varphi_0(x)|^2\, dx \to 0 \quad \text{as } y \to +0,$$

where $R_2(0,1) = \{0 < y < 1,\ -\infty < x < \infty\}$.

Remark. If $\varphi_0 \in W_2^{(r_1)}(R_1)$, r_1 is an integer, then $u \in S_2^{(r_1,r_2)}W[R_2(0,1)]$ for any integer r_2.

4. For $s = 2$ the solution of the boundary-value problem has the form

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-v_0 y) \left[\Phi_0(\lambda)(\cos v_0 y + \sin v_0 y) + \Phi_1(\lambda)\frac{\sin v_0 y}{v_0}\right] e^{i\lambda x}\, d\lambda, \tag{4,1}$$

where Φ_0, Φ_1 are the Fourier transforms of the functions φ_0 and φ_1, respectively,

$$v_0 = |\lambda|^{r/2}(1+\lambda^{2r})^{-1/4}.$$

Theorem 3. If the functions φ_0, φ_1 belong to the class $S_2^{(r_1)}H(R_1)$, $r_1 > 0$ $(S_2^{(r_1)}W(R_1)$, r_1 an integer), then the solution (4,1) belongs to the class $S_2^{(r_1,r_2)}H[R_2(0,1)]$ $(S_2^{(r_1,r_2)}W[R_2(0,1)])$ for any $r_2 > 0$.

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1. S. M. Nikol' skii, a) *Matem. sborn.*, **33** (75), 2, 261 (1958); b) *DAN*, **146**, No. 3, 543 (1962); c) *DAN*, **146**, No. 3, 767 (1962); d) Proceedings of the Second All-Union Conference on the Constructive Theory of Functions, Baku, 1962 (in press).

Note: Figure translations are in progress. See original paper for figures.

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