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**Abstract**

**Full Text**

## **THEORY OF ELASTICITY**

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# **ON THE LINEAR THEORY OF THREE-LAYER SHELLS WITH A RIGID CORE**

The general equations of equilibrium and free vibrations of thin three-layer shells of asymmetric construction with a rigid core compressed in the transverse direction are considered. In deriving the equations, the following assumptions are made. The load-bearing layers and the core have constant thicknesses and are made of different orthotropic materials. The core is regarded as a three-dimensional body, and it is assumed that the displacements of its points are approximated with sufficient accuracy by linear functions of the transverse coordinate. For the load-bearing layers the Kirchhoff–Love hypotheses are adopted. In contrast to the theory of shallow shells <sup>(1,2)</sup>, as well as in <sup>(3)</sup>, refined expressions are adopted for the angles of inclination of the normal to the original surface of the shell. It is assumed that the deformations remain elastic at all times.

The equations of motion are obtained from the variation of the total energy of the shell. In calculating the variation of the potential energy of the core, along with the energy of transverse shear, the energy of transverse compression was taken into account.

Taking as the original surface the middle surface of the core, we introduce on it coordinates  $x_1, x_2$ , referred to the lines of principal curvatures. The coordinate referred to the outer normal will be denoted by  $z$ . Let the lower index of a component  $i = 1, 2$  indicate that the given component belongs to the  $i$ -th direction, and the upper index  $j = 1, 2, 3$  to the upper layer, the lower layer, and the core, respectively. Partial differentiation with respect to  $x_i$  will be denoted by the lower index  $i$  after a comma.

We denote by  $h_1, h_2, 2c$  and  $\rho_1, \rho_2, \rho$  the thicknesses and specific densities, respectively, of the first and second load-bearing layers and of the core; by  $A_i$  and  $k_i$  the Lamé coefficients and the principal curvatures of the original surface; by  $H_i = A_i(1 + k_i z)$  the Lamé coefficients of a surface equidistant from the original one; by  $w$  and  $u_i$  the normal and tangential displacements of points of the original surface; by  $2a_i$  the absolute shear of the boundary surfaces of the core along the  $x_i$  axis; by  $v$  the absolute approach of the original and lower surfaces of the core; by  $q_i$  the tangential surface load; by  $p^+, p^-$  the normal surface loads applied respectively to the outer and inner sides of the composite shell; and by

$t$  the time.

The displacements will be written in the following form:

$$w^z = \begin{cases} w + v, & c \leq z \leq c + h_1, \\ w + \frac{z}{c}v, & -c \leq z \leq c, \\ w - v, & -c - h_2 \leq z \leq -c; \end{cases} \quad (1)$$

$$u_i^z = \begin{cases} u_i + a_i + (z - c) [(u_i + a_i)k_i - H_i^{-1}(w_{,i} + v_{,i})], & c \leq z \leq c + h_1, \\ u_i + \frac{z}{c}a_i, & -c \leq z \leq c, \\ u_i - a_i + (z + c) [(u_i - a_i)k_i - H_i^{-1}(w_{,i} - v_{,i})], & -h_2 - c \leq z \leq -c. \end{cases} \quad (2)$$

We introduce the specific forces and moments as follows:

$$\begin{aligned} T_{ij} &= T_{ij}^1 + T_{ij}^2 + T_{ij}^3; & M_{ij}^+ &= M_{ij}^1 + M_{ij}^2; \\ M_{ij}^- &= M_{ij}^1 - M_{ij}^2; & \widetilde{M}_{ij} &= M_{ij}^3 + c(T_{ij}^1 - T_{ij}^2); \end{aligned} \quad (3)$$

$$\begin{aligned} T_{ij}^1 &= \int_c^{c+h_1} \sigma_{ij}^1 dz; & T_{ij}^2 &= \int_{-c-h_2}^{-c} \sigma_{ij}^2 dz; & T_{ij}^3 &= \int_{-c}^{+c} \sigma_{ij}^3 dz; \\ M_{ij}^1 &= \int_c^{c+h_1} \sigma_{ij}^1 (z - c) dz; & M_{ij}^2 &= \int_{-c-h_2}^{-c} \sigma_{ij}^2 (z + c) dz; & M_{ij}^3 &= \int_{-c}^{+c} \sigma_{ij}^3 z dz; \end{aligned} \quad (4)$$

$$T_3 = \int_{-c}^c \sigma_3 dz; \quad Q_1 = \int_{-c}^c \sigma_{13} dz; \quad Q_2 = \int_{-c}^c \sigma_{23} dz.$$

The equations of motion have the form:

$$\begin{aligned}
& (A_2 T_{11})_{,1} - A_{2,1} T_{22} + A_1^{-1} (A_1^2 T_{12})_{,2} + k_1 (A_2 M_{11}^+)_{,1} - \\
& \quad - k_1 A_{2,1} M_{22}^+ + k_1 A_1^{-1} (A_1^2 M_{12}^+)_{,2} + (A_1 k_1 M_{12}^+)_{,2} + \\
& + k_2 A_{1,2} M_{12}^+ + k_1 A_1 A_2 Q_1 = -A_1 A_2 \left( q_1 - \Omega_{11} \frac{\partial^2 u_1}{\partial t^2} \right); \\
& -A_{1,2} T_{11} + (A_1 T_{22})_{,2} + A_2^{-1} (A_2^2 T_{12})_{,1} - k_2 A_{1,2} M_{11}^+ + \\
& \quad + k_2 (A_1 M_{22}^+)_{,2} + A_2^{-1} (A_2^2 M_{12}^+)_{,2} + (A_2 k_2 M_{12}^+)_{,1} + \\
& + k_1 A_{2,1} M_{12}^+ + k_2 A_1 A_2 Q_2 = -A_1 A_2 \left( q_2 - \Omega_{11} \frac{\partial^2 u_2}{\partial z^2} \right); \\
& (A_2 \widetilde{M}_{11})_{,1} - A_{2,1} \widetilde{M}_{22} + A_1^{-1} (A_1^2 \widetilde{M}_{12})_{,2} + (A_1 k_1 c M_{11}^-)_{,1} - k_1 c A_{2,1} M_{22}^- + \\
& \quad + k_1 c A_1^{-1} (A_1^2 M_{12}^-)_{,2} + (A_1 k_1 c M_{12}^-)_{,2} + k_2 c A_{1,2} M_{12}^- - A_1 A_2 Q_1 = 0; \\
& -A_{1,2} \widetilde{M}_{11} + (A_1 \widetilde{M}_{22})_{,2} + A_2^{-1} (A_2^2 \widetilde{M}_{12})_{,1} - k_1 c A_{1,2} M_{11}^- - k_1 c (A_1 M_{22}^-)_{,2} + \\
& \quad + k_1 c A_2^{-1} (A_2^2 M_{12}^-)_{,1} + (A_2 k_2 c M_{12}^-)_{,1} + k_1 c A_{2,1} M_{12}^- - A_1 A_2 Q_2 = 0; \tag{5} \\
& A_1 A_2 (k_1 T_{11} + k_2 T_{22}) - (A_1^{-1} (A_2 M_{11}^+)_{,1})_{,1} + (A_2^{-1} A_{1,2} M_{11}^+)_{,2} - \\
& \quad - (A_2^{-1} (A_1 M_{22}^+)_{,2})_{,2} + (A_1^{-1} A_{2,1} M_{22}^+)_{,1} - (A_2^{-2} (A_2^2 M_{12}^+)_{,1})_{,2} - \\
& \quad - (A_1^{-2} (A_1^2 M_{12}^+)_{,2})_{,1} - (A_2 Q_1)_{,1} - (A_1 Q_2)_{,2} = \\
& \quad = A_1 A_2 \left[ (p^+ + p^-) - \Omega_{11} \frac{\partial^2 w}{\partial z^2} + \Omega_{12} \frac{\partial^2 v}{\partial z^2} \right]; \\
& A_1 A_2 c^{-1} (k_1 \widetilde{M}_{11} + k_2 \widetilde{M}_{22}) + A_1 A_2 c^{-1} T_3 - (A_1^{-1} (A_2 M_{11}^-)_{,1})_{,1} + \\
& \quad + (A_2^{-1} A_{1,2} M_{11}^-)_{,2} - (A_2^{-1} (A_1 M_{22}^-)_{,2})_{,2} + (A_1^{-1} A_{2,1} M_{22}^-)_{,1} - \\
& \quad - (A_2^{-2} (A_2^2 M_{12}^-)_{,1})_{,2} - (A_1^{-2} (A_1^2 M_{12}^-)_{,2})_{,1} - \\
& \quad - \frac{2}{3} c [G_2 (A_2 v_{,1} A_1^{-1})_{,1} + G_3 (A_1 v_{,2} A_2^{-1})_{,2}] = \\
& \quad = A_1 A_2 \left[ (p^+ - p^-) + \Omega_{12} \frac{\partial^2 w}{\partial z^2} + \Omega_{22} \frac{\partial^2 v}{\partial z^2} \right].
\end{aligned}$$

The boundary conditions are determined from

$$\begin{aligned}
 \int_{\alpha_1}^{\alpha_2} \left\{ (T_{22} + k_2 M_{22}^+) \delta u_2 + (T_{12} + 2k_1 M_{12}^+) \delta u_1 + (\widetilde{M}_{22} + k_2 c M_{22}^-) \frac{\delta a_2}{c} \right. \\
 + (\widetilde{M}_{12} + 2k_1 c M_{12}^-) \frac{\delta a_1}{c} + (A_1 A_2)^{-1} [(A_1 M_{22}^+)_{,2} - A_{1,2} M_{11}^+ \\
 + 2(A_2 M_{12}^+)_{,1} + A_1 A_2 Q_2] \delta w + (A_1 A_2)^{-1} [(A_1 M_{22}^-)_{,2} \\
 - A_{1,2} M_{11}^- + 2(A_2 M_{12}^-)_{,1} + \frac{2}{3} c G_3 A_1 v_{,2}] \delta v \\
 \left. - M_{22}^+ A_2^{-1} \delta w_{,2} - M_{22}^- A_2^{-1} \delta v_{,2} \right\} A_1 dx_1 \Big|_{\beta_1}^{\beta_2} \\
 - 2M_{12}^+ \delta w \Big|_{\alpha_1 \beta_1}^{\alpha_2 \beta_2} - 2M_{12}^- \delta v \Big|_{\alpha_1 \beta_1}^{\alpha_2 \beta_2}; \quad (6)
 \end{aligned}$$

the integral corresponding to the other edge of the shell is obtained from (6) by the substitutions  $\alpha_1 \leftrightarrow \beta_1$ ,  $\alpha_2 \leftrightarrow \beta_2$ ,  $1 \leftrightarrow 2$ ,  $G_3 \rightarrow G_2$ . Here  $G_2$  and  $G_3$  are the shear moduli of the core material, respectively for the planes  $x_1 o z$  and  $x_2 o z$ ;  $\Omega_{11} = \rho_1 h_1 + \rho_2 h_2 + 2\rho c$ ;  $\Omega_{22} = \rho_1 h_1 + \rho_2 h_2 + \frac{2}{3}\rho c$ ;  $\Omega_{12} = \rho_1 h_1 - \rho_2 h_2$ .

We note that the system of equations presented is symmetric.

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*Note: Figure translations are in progress. See original paper for figures.*

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