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Abstract

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SOLUTION OF NON-SELF-ADJOINT EQUATIONS BY A VARIATIONAL METHOD*

(Presented by Academician S. L. Sobolev on 8 February 1963)

Let an equation be given

$$Au = f, \tag{1}$$

where A is some operator in a Hilbert space H; u, f are, respectively, the unknown and the given elements in H.

For the case when A is a self-adjoint positive definite operator, methods for investigating equations of the form (1), as well as methods for the effective construction of the solution, have been developed quite well. Of particular interest is the variational method for investigating equations developed in the works of K. Friedrichs (1), S. L. Sobolev (3), S. G. Mikhlin (5) and other authors, which, along with establishing the solvability of the equation and studying the properties of the solution, also poses the corresponding variational problem, often solved by the so-called "direct methods."

To generalize the variational method to the case of non-self-adjoint equations, we shall consider, along with the given operator A in equation (1), a certain auxiliary operator B. Such an approach is, to some extent, close to the method of K. Friedrichs *abc*, generalized by him to systems of differential equations (2), and also to the results of P. E. Sobolevskii (6), who considered, for establishing the solvability of equations, operators forming an acute angle. The direct application of the variational method to a particular class of non-self-adjoint differential equations is the subject of a paper by A. E. Martynyuk (4), and also of the recent paper by V. V. Petrishin (7), where the variational principle is generalized to a certain new class of equations in Hilbert space. In our paper (8) we defined the concept of a B-symmetric and B-positive operator A and established the properties of such operators. These results, as well as the notation of (8), will be used below.

We shall assume that the operators A and B possess the following properties:
1) $(Au, Bv) = (Bu, Av)$ for all elements $u, v \in D(A, B) = D(A) \cap D(B)$ (the property of B-symmetry of A);

2) $(Au, Bu) > 0$ for $u \neq 0$; from the condition $(Au, Bu) \to 0$ it follows that $\|u\| \to 0$ for $u \in D(A, B)$ (the property of B-positivity of A).

For such operators one can construct the B-extensions, in the sense of K. Friedrichs, $\mathcal{A}$ and $\mathcal{B}$, defined in (^{8}), and one can introduce the complete space $F_{A\mathcal{B}}$ with scalar product:

$$[uv]_{A\mathcal{B}} = (\mathcal{A}u, \mathcal{B}v)$$

and the corresponding norm

$$\|u\|_{A\mathcal{B}} = (\mathcal{A}u, \mathcal{B}u)^{1/2},$$

which is a generalization of the space introduced earlier by K. Friedrichs (^{1}). Instead of equation (1) we shall consider the equation

$$\mathcal{A}u = f \tag{2}$$

and the functional

$$\mathfrak{D}_f(u) = (\mathcal{A}u, \mathcal{B}u) - (\mathcal{B}u, f) - (f, \mathcal{B}u). \tag{3}$$

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* The work was reported on 10 June 1961 at the Fourth All-Union Mathematical Congress in Leningrad.

Obtaining the solution of equation (2) will be called problem $\mathcal{A}_f$. Finding the element that realizes the minimum of the functional (3) will be called, respectively, problem $\mathcal{D}_f$ or the variational problem. If every solution of problem $\mathcal{A}_f$ is also a solution of problem $\mathcal{D}_f$ and, conversely, every solution of problem $\mathcal{D}_f$ is also a solution of $\mathcal{A}_f$, then such problems will be called equivalent.

We shall also agree that an element $u_0 \in \mathfrak{F}_{A\mathcal{B}}$ will be called a solution of the variational problem $\mathcal{D}_f$ if in $D(\mathcal{A}, \mathcal{B}) = D(\mathcal{A}) \cap D(\mathcal{B})$ there exists a sequence $\{u_k\}$ such that, simultaneously as $k \to \infty$, the relations $\mathcal{D}_f(u_k) \to \mathcal{D}_f(u_0) = d$ and $\|u_k - u_0\| \to 0$ hold, where d is the exact lower bound of the functional $\mathcal{D}_f(u)$.

For non-self-adjoint equations the following is true.

Theorem 1. *If the operator is B-positive, then problem $\mathcal{A}_f$ and problem $\mathcal{D}_f$ are equivalent.*

The consideration of B-positive operators is quite natural when using the variational method; namely, the following holds.

Theorem 2. *For given operators A and B and an element $f_0 \in H$, the solution of the variational problem $\mathcal{D}_{f_0}$ exists and is unique if and only if: 1) the operator A is B-positive; 2) the functional (Bu, f_0) is linear on $\mathfrak{F}_{A\mathcal{B}}$.*

This theorem gives necessary and sufficient conditions for the unique solvability of the variational problem $\mathcal{D}_{f_0}$, and, by virtue of Theorem 1, also of problem $\mathcal{A}_{f_0}$. The element f_0 is fixed; if, however, the functional (Bu, f_0) is linear on $\mathfrak{F}_{A\mathcal{B}}$ for every $f \in H$, then the problems $\mathcal{D}_f$ and $\mathcal{A}_f$ are solvable respectively for every f. Sometimes, however, it is easier to establish another condition for solvability of equation (2) and problem $\mathcal{D}_f$ than the condition in Theorem 2.

Theorem 3. *If, for a given operator A, there are an operator B and a number $\gamma > 0$ such that the operator $A - \gamma B$ is B-positive, then problem $\mathcal{D}_f$ and problem $\mathcal{A}_f$ are uniquely solvable for every element $f \in H$.*

The proof of the theorem follows elementarily from Theorem 2.

Up to now the existence of an operator B with the indicated properties has been assumed. Conditions sufficient for the existence of the operator B are given by Theorem 4.

Theorem 4. *If $D(A)$ and $R(A)$ are dense in H and, for some nonzero element $f_0 \in H$, the solution of the equation*

$$Au = f_0$$

exists and is unique, then:

1) there exists an operator B such that the operator A will be B-positive;

2) the operator B can be represented in the form:

$$B = (A^*)^{-1}C,$$

where C is an arbitrary I-positive operator with $D(C) \supseteq D(A)$.

This theorem guarantees the existence of an operator B in the case where equation (2) is solvable. The actual finding of B presents no fundamental difficulties and in many cases is done quite elementarily.

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Note: Figure translations are in progress. See original paper for figures.

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